We consider a band based on an ellipse $x^2/a^2+y^2/b^2=1$ with hight $|\cos\theta|$. Calculate the area.

The integral is given by (Using $z=\sqrt{|b^2-a^2|}/bu$)

$$\begin{aligned}
\int_0^{2\pi} |\cos(\theta)|\sqrt{a^2\sin^2(\theta)+b^2\cos^2(\theta)}d\theta &= 4\int_0^{\frac{\pi}{2}} \cos(\theta)\sqrt{b^2+(a^2-b^2)\sin^2(\theta)}d\theta \\
&= 4\int_0^1 \sqrt{b^2+(a^2-b^2)u^2}du \\
&= 4b\int_0^1 \sqrt{1+(\frac{a^2-b^2}{b^2})u^2}du \\
&= \frac{4b^2}{\sqrt{|a^2-b^2|}}\int_0^{\frac{\sqrt{|a^2-b^2}}{b}} \sqrt{1\pm z^2}dz\ .
\end{aligned}$$

Here we have $+z$ if $a^2 \geq b^2$ and $-z$ else.

Let us discuss $\int\sqrt{1+z^2}dz$:

$$\begin{aligned}
\int\sqrt{1+z^2}dz &= \int\sqrt{1+z^2}1dz \\
&= z\sqrt{1+z^2}-\int\frac{z^2}{\sqrt{1+z^2}}dz \\
&= z\sqrt{1+z^2}-\int\frac{1+z^2}{\sqrt{1+z^2}}dz+\int\frac{1}{\sqrt{1+z^2}}dz \\
&= z\sqrt{1+z^2}-\int\sqrt{1+z^2}dz+\sinh^{-1}(z)\ .
\end{aligned}$$

Therefore

$$\int\sqrt{1+z^2}dz \;=\; \frac{z\sqrt{1+z^2}+\sinh^{-1}(z)}{2}\ .$$

Let us discuss $\int\sqrt{1-z^2}dz$:

$$\begin{aligned}
\int\sqrt{1-z^2}dz &= \int\sqrt{1-z^2}1dz \\
&= z\sqrt{1-z^2}+\int\frac{z^2}{\sqrt{1-z^2}}dz \\
&= z\sqrt{1+z^2}-\int\frac{1-z^2}{\sqrt{1-z^2}}dz+\int\frac{1}{\sqrt{1-z^2}}dz \\
&= z\sqrt{1+z^2}-\int\sqrt{1+z^2}dz+\sin^{-1}(z)\ .
\end{aligned}$$

Therefore

$$\int\sqrt{1+z^2}dz \;=\; \frac{z\sqrt{1-z^2}+\sin^{-1}(z)}{2}\ .$$