Parametric curves and related problems

1) Consider the curve C given by

$$x^3 + xy^2 + y^3 \;=\; 3$$

Find the intersection of the x-axis of the tangent line of C at $(1, 1)$.

Sol.: We assume $\gamma(t) = (x(t), y(t))$ lies on the curve. Then

$$\frac{d}{dt}x(t)^3 + x(t)y(t)^2 + y(t)^3 \;=\; 3$$

and hence

$$3x^2\frac{dx}{dt} + \frac{dx}{dt}y + 2xy\frac{dy}{dt} + 3y\frac{dy}{dt} \;=\; 0\,.$$

For $x = y = 1$ this gives

$$4\frac{dx}{dt} + 5\frac{dy}{dt} \;=\; 0$$

Hence

$$\frac{dy}{dx} \;=\; -\frac{4}{5}$$

is the slope. The line equation is

$$y \;=\; -\frac{4}{5}x + C$$

Since $(1, 1)$ is on the line we get

$$1 \;=\; -4/5 + C$$

and hence $C = 9/5$. The intersection with the x-axis occurs for $y = 0$, i.e.

$$0 \;=\; -\frac{4}{5}x + \frac{9}{5}\,.$$

This means $x = 9/4$.

2) Let $\gamma(t) = r(t)(\cos(\theta(t)), \sin(\theta(t))$ such that r is increasing and

$$\theta(t) \;=\; c + d\ln r(t)\,.$$

Show that

$$|\gamma'(t)| \;=\; \sqrt{1+d^2}r'(t)\,.$$

Sol.: We use the formula for parametric curves in polar coordinates

$$|\gamma'(t)| \;=\; \sqrt{(r'(t))^2 + (r(t)\theta'(t))^2}\,.$$

Note that

$$r(t)\theta'(t) \;=\; r(t)\frac{dr'(t)}{r(t)} \;=\; dr'(t)\,.$$

Thus

$$\sqrt{(r'(t))^2 + (r(t)\theta'(t))^2} \;=\; r'(t)\sqrt{1+d^2}\,,$$

because $r'(t) \ \geq \ 0$.

3) Find the length of $\gamma(t) \ = \ t^2(\cos(1+\ln t), \sin(1+\ln t)$, $0 \leq t \leq 1$.

Sol.: We have $\theta'(t) \ = \ \frac{1}{t}$ and hence

$$r(t)\theta'(t) \ = \ t^2\frac{1}{t} \ = \ t\,.$$

Thus

$$|\sqrt{(r'(t))^2 + (r(t)\theta'(t))^2} \ = \ t\sqrt{4+1} \ = \ \sqrt{5}t\,.$$

This gives

$$L(\gamma) \ = \ \int_0^1 \sqrt{5}tdt \ = \ \frac{\sqrt{5}}{2}\,.$$

4) Find the length of $\gamma(t) \ = \ (\frac{t^3}{3}, \frac{t^4}{4})$, $0 \leq t \leq 1$.

Sol.: We have

$$\gamma'(t) \ = \ (t^2, t^3)$$

and

$$|\gamma'(t)| \ = \ \sqrt{t^4 + t^6} \ = \ t^2\sqrt{1+t^2}\,.$$

Therefore we shall compute

$$\int \sqrt{1+t^2}t^2dt = \int \sqrt{1+t^2}(t^2+1)dt - \int \sqrt{1+t^2}dt$$

We use $t = \sinh(x)$ for both:

$$\int \sqrt{1+t^2}dt \ = \ \int \cosh^2(x)dx \ = \ \frac{\cosh x \sinh x + x}{2}\,.$$

And also

$$\begin{aligned} \int \sqrt{1+t^2}(t^2+1)dt \ &= \ \int \cosh^4(x)dx \\ &= \frac{\cosh^3 x \sinh x}{4} + \frac{1}{4}\int \cosh^2 dx \\ &= \frac{\cosh^3 x \sinh x}{4} + \frac{\cosh x \sinh x + x}{4}\,. \end{aligned}$$

This gives ($t = \sinh x$, $1 + \sinh^2 x = \cosh(x)$)

$$\int \sqrt{1+t^2}(t^2+1)dt - \int \sqrt{1+t^2}dt \ = \ \frac{\cosh^3 x \sinh x}{4} - \frac{\cosh x \sinh x + x}{4}$$

$$= = \frac{(1+t^2)^{3/2}t}{4} - \frac{(1+t^2)^{1/2}t + \sinh^{-1} t}{4}\,.$$

Thus for the length we get

$$\int_0^1 \sqrt{1+t^2}t^2dt \ = \ \frac{2^{3/2}}{4} - \frac{2^{1/2} + \sinh^{-1}(1)}{4}\,.$$

5) Find the volume of revolution around the x-axis for $\gamma(t) = (t, \sqrt{1+t^2})$, $0 \le t \le 1$. Also find the area of the body of revolution around the x-axis.

Sol.: The volume is given by

$$vol = \pi \int_0^1 y^2 dx = \pi \int_0^1 y(t)^2 dt = \pi \int_0^1 \sqrt{1+t^2} dt .$$

Using $t = \tan(u)$ we get

$$\int \sqrt{1+t^2} dt = \int \sec^3(u) du .$$

The reduction formula for $\sec^3$ is

$$\int \sec^3(u) du = \frac{1}{2}(\tan u \sec u + \int \sec u du) = \frac{1}{2}(\tan u \sec u + \ln(\sec(u) + \tan(u)) .$$

Thus

$$\pi \int_0^1 \sqrt{1+t^2} dt = \frac{\pi}{2}(\tan(1)\sec(1) + \ln(\sec(1) + \tan(1)) .$$

The area is given by

$$Area = \int_{x=0}^1 2\pi y ds$$

where

$$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} dt .$$

This gives with $dy/dt = \frac{t}{\sqrt{1+t^2}}$, $v = \sqrt{2}t$, $dv = \sqrt{2}dt$

$$\begin{aligned} Area &= 2\pi \int_0^1 \sqrt{1+t^2} \sqrt{1 + \frac{t^2}{1+t^2}} dt \\ &= 2\pi \int_0^1 \sqrt{1+2t^2} dt \\ &= 2^{1/2}\pi \int_0^{\sqrt{2}} \sqrt{1+v^2} du \\ &= 2^{1/2}\pi \frac{1}{2}((\tan\sqrt{2}\sec\sqrt{2} + \ln(\sec\sqrt{2} + \tan\sqrt{2})) . \end{aligned}$$

6) Find the area inside the cardioid $r = 1 - \sin\theta$.

Sol.: We have to consider

$$B = \{r(\cos(\theta), \sin(\theta)) : 0 \le \theta \le 2\pi, 0 \le r \le 1 - \sin(\theta)\}$$

By the change of variable formula $dxy = r dr d\theta$ we get

$$\begin{aligned} Area(B) &= \int_0^{2\pi} \int_0^{1-\sin(\theta)} r dr d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1 - \sin(\theta))^2 d\theta \end{aligned}$$

$$= \frac{1}{2}(2\pi - 2\int_0^{2\pi} \sin(\theta)d\theta + \int_0^{2\pi} \sin^2(\theta)d\theta)\ .$$

The integral in the middle is 0 (as many sin above 0 then below). Furthermore

$$\int_0^{2\pi} \sin^2(\theta)d\theta \ = \ \frac{1}{2}\int_0^{2\pi} \sin^2(\theta) + \cos^2(\theta)d\theta \ = \ \pi$$

by change of variable. Hence

$$Area(B) \ = \ \frac{\pi}{2}\ .$$