Homework 5-Math 231 h

Due date: October 11. Submission in pairs.

(1) page 768 no 13, 14* (ignore linearly independent-just find two solutions).
(2) page 768 no 20,21,22*.
(3) page 769 no 5, 11, 15, 12*.
(4) page 770 no 35, 37, 38*. no 36 is extra credit.

1. An example using power series

We consider

$$f'' = f$$

with $f(0) = 1,\ f'(0) = -1$.
The point is to try

$$f(x) = \sum_{k \geq 0} a_k x^k .$$

Then

$$f'(x) = \sum_{k \geq 1} a_k k x^{k-1} .$$

and

$$f''(x) = \sum_{k \geq 2} a_k k(k-1)x^{k-2} = \sum_{k \geq 0} a_{k+2}(k+2)(k+1)x^k .$$

How to explain the last step? The smallest exponent in the first sum is $x^{2-2} = x^0$. SO we have to start with $k = 0$. Obviously, the k in a_k is two bigger than the exponent. So a_{k+2} corresponds to x^k,...
The equation $f'' = f$ entails

$$\sum_{k \geq 0} a_k x^k = \sum_{k \geq 0} a_{k+2}(k+2)(k+1)x^k .$$

For this to work we need

$$a_k = (k+2)(k+1)a_{k+2} .$$

Also

$$a_0 = f(0) = 1 \ ,\ a_1 = f'(0) = -1 .$$

This gives

$$a_2 = \frac{1}{2}a_0 = \frac{1}{2}$$

$$a_3 = \frac{1}{3}\frac{1}{2}a_1 = -\frac{1}{3!} .$$

In general

$$a_k = \frac{(-1)^k}{k!} .$$

The solution is

$$f(x) = \sum_{k \lg 0} a_k x^k = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} - \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} .$$

As seen today all this calculation would be useless without having convergence.

$$\lim_k |a_k|^{1/k} \;=\; \lim_k \left(\frac{1}{k!}\right)^{\frac{1}{k}} \;=\; 0\,.$$

Let us show this: Let $\varepsilon > 0$. Then we may choose k_0 such that $k_0 > \frac{1}{\varepsilon^2}$. Since

$$\lim_{k\to\infty} \frac{(k-k_0)\ln k_0}{k} \;=\; \ln k_0$$

we can find k_1 such that for $k > k_1$ we have

$$\frac{(k-k_0)\ln k_0}{k} > \frac{\ln k_0}{2}\,.$$

This implies

$$\frac{1}{k}\sum_{j=1}^{k} \ln j \;\geq\; \frac{(k-k_0)\ln k_0}{k} > \frac{\ln k_0}{2}$$

for $k > k_1$. Taking exponentials we get

$$(k!)^{1/k} \;\geq\; e^{\frac{\ln k_0}{2}} \;=\; \sqrt{k_0}\,.$$

This gives

$$0 < (k!)^{1/k} \;\leq\; \frac{1}{\sqrt{k_0}} < \varepsilon$$

for all $k > k_1$. Thus

$$\lim_k (k!)^{-1/k} \;=\; 0$$

and hence

$$\frac{1}{R} \;=\; \lim_k (k!)^{-1/k}\,.$$

This means $R = \infty$. This is good! Our solution exists for all x and the formula's

$$f'(x) \;=\; \sum_k k a_k x^{k-1}$$

$$f''(x) \;=\; \sum_k k(k-1) a_k x^{k-2}$$

are indeed valid (not so with the example from class).

Conclusion: We have indeed found a solution.

Remark: In the book I found

$$\cosh(x) \;=\; \frac{e^x + e^{-x}}{2} \;=\; \frac{1}{2}\sum_k \frac{x^k}{k!} + \frac{1}{2}\sum_k (-1)^k \frac{x^k}{k!}$$

$$= \sum_{k \text{even}} \frac{x^k}{k!} \;=\; \sum_{k\,\geq\,0} \frac{x^{2k}}{(2k)!}\,.$$

Similarly,

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \; = \; \frac{1}{2}\sum_k \frac{x^k}{k!} - \frac{1}{2}\sum_k (-1)^k \frac{x^k}{k!}$$

$$= \sum_{k \mathrm{odd}} \frac{x^k}{k!} \; = \; \sum_{k \,\geq\, 0} \frac{x^{2k+1}}{(2k+1)!} \; .$$

Hence our solution is

$$f(x) \; = \; \cosh(x) - \sinh(x) \; .$$

Let's check! $f(0) = 1$, $f'(0) = -1$ and

$$f''(x) \; = \; \cosh''(x) - \sinh''(x) \; = \; \cosh(x) - \sinh(x) \; .$$

Yes, indeed, $f(x) = \cosh(x) - \sinh(x)$ is a solution.