Limits

1. Limits of sequences

1) $\lim_n \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1-\frac{1}{n}}}$,

Sol.: The function $f(x) = \sqrt{x}$ is continuous at 1. Hence $\lim_{x\to 0} \sqrt{1+x} = 1$. Since $\lim_n \frac{1}{n} = 0$, we may apply the limit laws and get

$$\lim_n \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1-\frac{1}{n}}} \;=\; \frac{\lim_n \sqrt{1+\frac{1}{n}}}{\lim_n \sqrt{1-\frac{1}{n}}} \;=\; \frac{1}{1} \;=\; 1\,.$$

2) $\lim_n n(\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}})$

Sol.: The function $f(x) = \sqrt{x}$ is differentiable. Hence

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}\,.$$

For $x = 1$ we get

$$\lim_{n\to\infty} \frac{f(x+\frac{1}{n})-f(x-\frac{1}{n})}{2\frac{1}{n}} \;=\; f'(1) \;=\; 1\,.$$

Hence

$$\lim_n n(\sqrt{1+\frac{1}{n}} - \sqrt{1-\frac{1}{n}}) \;=\; \frac{1}{2}\,.$$

3) $\lim_n (1+\frac{3}{n})^{n^2}$

Sol.: We take the logarithm

$$\lim_n n^2 \ln(1+\frac{3}{n}) \;=\; \lim_n \frac{\ln(1+\frac{3}{n})}{\frac{1}{n^2}}\,.$$

By l'Hopital

$$\lim_{x\to 0} \frac{\ln(1+3x)}{x^2} \;=\; \lim_{x\to 0} \frac{3/(1+3x)}{2x} \;=\; \lim_{x\to 0} \frac{3}{2(1+3x)}\frac{1}{x} \;=\; \infty\,.$$

Since $e^\infty = \infty$ we deduce

$$\lim_n (1+\frac{3}{n})^{n^2} \;=\; \infty\,.$$

4) $\lim_n (1+\frac{3}{n^2})^n$,

Sol.: Same procedure.

$$\lim_n n \ln(1+\frac{3}{n^2})$$

and y l'Hopital

$$\lim_{x\to 0} \frac{\ln(1+3x^2)}{x} \;=\; \lim_{x\to 0} \frac{\frac{6x}{1+3x^2}}{1} \;=\; 0\,.$$

Hence

$$\lim_n (1+\frac{3}{n^2})^n \;=\; e^0 \;=\; 1\ .$$

5) $\lim_n \frac{(1+\frac{1}{n})^{n^2}}{e^n}$

Sol.: We take the logarithm:

$$\lim_n (n^2 \ln(1+\frac{1}{n}) - n) = \lim_n n(n\ln(1+\frac{1}{n}) - 1)\ .$$

Note that $\lim_{x\to 0} \frac{\ln(1+x)}{x} \;=\; 1$. Thus we may use l'Hopital

$$\begin{aligned}\lim_{x\to 0} \frac{\frac{\ln(1+x)}{x} - 1}{x} &= \lim_{x\to 0} \frac{\frac{\frac{1}{1+x}x - \ln(1+x)}{x^2}}{1} \;=\; \lim_{x\to 0} \frac{\frac{1}{1+x}x - \ln(1+x)}{x^2} \\ &= \lim_{x\to 0} \frac{\frac{1}{1+x} - \frac{x}{(1+x)^2} - \frac{1}{(1+x)}}{2x} \;=\; \lim_{x\to 0} -\frac{1}{2(1+x)^2} \;=\; -\frac{1}{2}\ .\end{aligned}$$

Hence $\lim_n \frac{(1+\frac{1}{n})^{n^2}}{e^n} \;=\; e^{-1/2}$.

6) $\lim_{x\to 0} \frac{(1+x)^{1/3}-1}{(1+x)^{1/2}-1}$

Sol.: The function $f(x) = (1+x)^{1/3}$ is differentiable. Hence

$$\lim_{x\to 0} \frac{f(x)-1}{x} \;=\; f'(1) \;=\; \frac{1}{3}\ .$$

Similarly,

$$\lim_{x\to 0} \frac{(1+x)^{1/2}-1}{x} \;=\; \frac{1}{2}\ .$$

Taking the ratio gives

$$\lim_{x\to 0} \frac{(1+x)^{1/3}-1}{(1+x)^{1/2}-1} \;=\; \frac{\frac{1}{3}}{\frac{1}{2}} \;=\; \frac{2}{3}\ .$$

Problem for the definition

a) Find n_ε such that

$$-\varepsilon < \frac{n^2+1}{n^2-1} - 1 < \varepsilon$$

holds for all natural numbers $n > n_\varepsilon$.

Sol.: We observe that

$$\frac{n^2+1}{n^2-1} - 1 \;=\; \frac{n^2+1-(n^2-1)}{n^2-1} \;=\; \frac{2}{n^2-1}\ .$$

This expression is > 0 for $n > 1$. Moreover,

$$\frac{2}{n^2-1} < \varepsilon$$

holds for

$$\sqrt{\frac{2}{\varepsilon}+1} < n\ .$$

Thus n_ε is the smallest integer bigger or equal to $\sqrt{\frac{2}{\varepsilon}+1}$.

b) Use the definition to show that

$$\lim_n \frac{n^2+1}{n^2-1} \;=\; 1\,.$$

Sol.: a) is the definition.

2. series

1) $\sum_{k\,\geq\,1} \frac{k^2}{k^4-3}$,

Sol.: We observe that for $k \;\geq\; 2$ the coefficients are positive. Let $a_k = \frac{k^2}{k^4-3}$ and $b_k = \frac{1}{k^2}$. Then

$$\lim_k \frac{a_k}{b_k} \;=\; 1\,.$$

We know that

$$\sum_k \frac{1}{k^2}$$

converges because

$$\int_1^\infty \frac{1}{x^2} \;=\; 1$$

is finite. Hence the limit is finite.

2) $\sum_k(\sqrt{k+1}-\sqrt{k})$,

Sol.: Clearly, $a_k \;=\; \sqrt{k+1}-\sqrt{k} \;\geq\; 0$. We claim that for $b_k \;=\; \frac{1}{\sqrt{k}}$ we have $\lim_k \frac{b_k}{a_k} = 1/2$. Indeed,

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \;=\; \frac{1}{2}\,.$$

Therefore

$$\begin{aligned}\lim_k \frac{\sqrt{k+1}-\sqrt{k}}{\frac{1}{k}} &= \lim_k \frac{\sqrt{k}(\sqrt{1+\frac{1}{k}}-\sqrt{1})}{\frac{1}{\sqrt{k}}}\\ &= \lim_k \frac{\sqrt{1+\frac{1}{k}}-\sqrt{1})}{\frac{1}{k}} \;=\; \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \;=\; \frac{1}{2}\,.\end{aligned}$$

Since $\int_1^\infty \frac{dx}{\sqrt{x}} \;=\; \infty$, we get

$$\sum_k(\sqrt{k+1}-\sqrt{k}) \;=\; \infty\,.$$

4) $\sum_k \frac{(\sqrt{k+1}-\sqrt{k})}{k}$,

Sol: Same trick. Let $b_k = k^{3/2}$. As above

$$\lim_k \frac{\frac{(\sqrt{k+1}-\sqrt{k})}{k}}{k^{3/2}} \;=\; \frac{1}{2}\,.$$

Now, $\int_1^\infty x^{-3/2} dx$ is finite. Thus

$$\sum_k \frac{(\sqrt{k+1}-\sqrt{k})}{k}$$

is finite.

3) $\sum_k (-1)^k(\sqrt{k+1}-\sqrt{k})$,

Sol.: Let us show that

$$a_k \;=\; (\sqrt{k+1}-\sqrt{k})$$

is decreasing. This means

$$a_{k+1} \le a_k?$$

Indeed,

$$a_k - a_{k+1} \;=\; \sqrt{k+2}-\sqrt{k+1}-(\sqrt{k+1}-\sqrt{k}) \;=\; \sqrt{k+2}-\sqrt{k} \;\ge\; 0\,.$$

Thus the criterion for alternating series applies, because $\lim_k a_k = 0$.

5) $\sum_k (-1)^k((k+1)^{1/4}-k^{1/4})$,

As above we see that $a_k = ((k+1)^{1/4}-k^{1/4})$ is decreasing. In order to show that this sequence converges to 0 we observe that

$$(k+1)^{1/4}-k^{1/4} \;=\; k^{1/4}[(1+\frac{1}{k})^{1/4}-1]\,.$$

Since $f(x) = x^{1/4}$ is differentiable, we get

$$\lim_{x\to 0} \frac{(1+x)^{1/4}-1}{x} \;=\; \frac{1}{4}\,.$$

This means $\lim_k k((1+1/k)^{1/4}-1) = 1/4$ and hence

$$\lim_k k^{1/4}[(1+\frac{1}{k})^{1/4}-1] \;=\; \lim_k \frac{k[(1+\frac{1}{k})^{1/4}-1]}{k^{3/4}} \le 0\,.$$

So yes, it converges.

6) $\sum_k x^k \frac{\ln k}{(\ln(k+1))^2}$ (for $x > 1$, for $x < 1$).

Sol.: We need the radius of convergence for

$$a_k \;=\; \frac{\ln k}{(\ln(k+1))^2}\,.$$

Then

$$\lim_k \frac{a_k}{a_{k+1}} \;=\; \lim_k \frac{\ln k \ln^2(k+1)}{\ln(k+1)\ln^2(k+2)} \;=\; \lim_k \frac{\ln k \ln(k+1)}{\ln^2(k+2)}\,.$$

Using l'Hopital, we know that $\lim_k \frac{\ln k}{\ln(k+1)} = 1$ and $\lim_k \frac{\ln k}{\ln(k+2)} = 1$. Thus

$$\lim_k \frac{a_k}{a_{k+1}} \;=\; 1\,.$$

Thus for $x > 1$ we have divergence and for $x < 1$ we have convergence. For $x = 1$ we don't have convergence. Indeed, use the integral comparison test this is equivalent to

$$\sum_k \frac{1}{\ln k}$$

But

$$\int_e^\infty \frac{dx}{\ln x} \;=\; \infty \; .$$

7) $\ln_n \frac{\sum_{k=1}^n \frac{1}{k}}{\ln k}$.

Sol.: By integral comparison

$$\ln k \;\leq\; \sum_{k=1}^n \frac{1}{k} \;\leq\; 1 + \ln(k-1) \; .$$

This means for $a_k = \sum_{k=1}^n \frac{1}{k}$ and $b_k = \ln k$ we have

$$1 \;\leq\; \frac{a_k}{b_k} \;\leq\; \frac{1}{\ln k} + \frac{\ln(k-1)}{\ln k}$$

Both sides go to 1. By squeeze, we get

$$\lim_k \frac{a_k}{b_k} \;=\; 1 \; .$$