Solutions for quiz

1) We have

$$\lim_k (1 + \frac{1}{k^2})^k \;=\; 1$$

Indeed, taking the logarithm we know by l'Hoptial

$$\lim_k \frac{\ln(1+\frac{1}{k^2})}{\frac{1}{k}} \;=\; \lim_{x\to 0} \frac{\ln(1+x^2)}{x} \lim_{x\to 0} \frac{\frac{2x}{1+x^2}}{1} \;=\; 0$$

2) $\sum_n \frac{n^4+n^3+n}{n^4+3n^2-\pi}$ can be understood using the comparison principle for positive sequences:

$$\lim_n \frac{n^4+n^3+n}{n^4+3n^2-\pi} \;=\; 1$$

means that $\sum_n \frac{n^4+n^3+n}{n^4+3n^2-\pi}$ converges if and only if $\sum_n 1$ converges. The latter diverges, so does $\sum_n \frac{n^4+n^3+n}{n^4+3n^2-\pi}$

3) For $S = \sum_k (-1)^k a_k$ with a_k strictly decreasing we have

$$|S - \sum_{k=1}^n (-1)^k a_k| < a_n$$

For $a_k = 1/\sqrt{k}$ we find

$$|S - \sum_{k=1}^n (-1)^k a_k| < \frac{1}{\sqrt{n}} \, .$$

For $\sqrt{n} \;\geq\; 100$, i.e. $n \;\geq\; 10000$ we get

$$|S - \sum_{k=1}^n (-1)^k a_k| < 0.01 \, .$$

Don't try by hand.

4) $f'' + 2f \;=\; 3f'$ with $f(0) = 1$ and $f'(0) = 2$. We use power series and get

$$\sum_k [(k+2)(k+1)a_{k+2} + 2a_k]x^k \;=\; \sum_k 3(k+1)a_{k+1}x^k$$

Lets try our standard trick $a_k \;=\; \frac{b_k}{k!}$. Then we need

$$b_{k+2} + 2b_k \;=\; 3b_{k+1}$$

and $b_0 = a_0 = 1$, $b_1 = a_1 = 2$. Thus $b_0 = 1$, $b_1 = 2$,

$$b_2 \;=\; 6 - 2 \;=\; 4 \, ,$$

$$b_3 \;=\; 3 \times 4 - 2 \times 2 \;=\; 8 \, .$$

Let's try $b_k = 2^k$. Then

$$2^{k+2} + 22^k \;=\; 62^k \;=\; 32^{k+1} \, .$$

That works and we get

$$a_k = \frac{2^k}{k!}\,.$$

We know that $\lim_k (k!)^{1/k} = \infty$. Since $(2^k)^{1/k} = 2$. We find $R = \infty$ for

$$f(x) = \sum_k \frac{2^k}{k!} x^k\,.$$