How to use integrals for summation

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$$\ln(n+1) \;\leq\; \sum_{k=1}^{n} \frac{1}{k} \;\leq\; 1 + \ln n \; .$$

Proof. We use the definition of the integral with upper and lower sums. The lower sums for $\int_1^n \frac{dx}{x}$ with mesh size 1 is given by

$$L \;=\; \frac{1}{2} + \cdots + \frac{1}{n} \; .$$

Hence

$$\ln n \;\geq\; L \;\geq\; \frac{1}{2} + \cdots + \frac{1}{n}$$

This gives $\sum_{k=1}^{n} \frac{1}{k} \;\leq\; 1 + \ln n \; .$

As for the upper sums for the same integral we find

$$1 + \frac{1}{2} + \cdots \frac{1}{n-1} \ .$$

Thus

$$\sum_{k=1}^{n-1} \frac{1}{k} \ \geq \ \ln n \ .$$

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Lemma 2. *Let* $1\leq m\leq n$ *and* $a=\frac{n}{m}$*. Then*

$$0\ \leq\ \ln\frac{n}{m}-\sum_{j=m+1}^{n}\frac{1}{j}\ \leq\ \frac{1}{m}\ .$$

Proof. We use the mesh-size $\frac{1}{m}$. Thus for the lower bound we find some thing like this

$$L = \frac{1}{m}[\frac{1}{1+\frac{1}{m}} + \frac{1}{1+\frac{2}{m}} + \cdots +$$

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How does it end? There is an integer so that $1 + \frac{k}{m} = a = \frac{n}{m}$. Indeed, by multiplying with m we see that $k = n - m$ does the job. At the last step we find

$$\frac{1}{m}\frac{1}{a}\,.$$

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What do we find for the upper sums?

$$U \;=\; \frac{1}{m}[1 + \frac{1}{1+\frac{1}{m}} + \cdots + \frac{1}{1+\frac{k-1}{m}}]$$

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The interesting point is to calculate the difference:

$$\ln a - \sum_{j=m+1}^{n} \frac{1}{j} \leq U - L = \frac{1}{m} - \frac{1}{am} \leq \frac{1}{m}$$

Theorem 3. *Let* $n \geq m$. *Then*

$$|\sum_{j=1}^{n} \frac{1}{j} - \ln n - (\sum_{j=1}^{m} \frac{1}{j} - \ln m)| \leq \frac{1}{m} .$$

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What does it mean? Let $m = 5$. Then we calculate

$$a(5) = \sum_{j=1}^{5} \frac{1}{j} - \ln 5 = \frac{137}{60} - \ln 5 \simeq 0.674$$

in advance. Then

$$\sum_{j=1}^{n} \frac{1}{j} \sim \ln n + a(5)$$

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holds with an accuracy of $1/5 = 0.2$. Also for very large numbers, say $n = 10^6$! For an accuracy of $1/10$ we have to calculate

$$a(10) \simeq 0.626 .$$