(1) (25P) Let $A \subset \mathbb{R}$ be a closed bounded set. Show that

$$\sup A \in A \, .$$

(2) (30P) We consider our favorite metric space $(\mathbb{Z}, d_q)$ with the q-adic metric. For $x \in \mathbb{Z}$, we recall $r_k(x)$ to be the remainder with respect to q^k, i.e.

$$x \;=\; a_k q^k + r_k(x) \; , \; 0 \leq r_k(x) < q^k$$

for some $a_k \in \mathbb{Z}$. Show that the map $u : \mathbb{Z}_q \to \mathbb{R}$ (usual metric for $\mathbb{R}$) defined by

$$u(x) \;=\; \sum_{k=1}^{\infty} r_k(x) q^{-2k}$$

is Lipschitz.

(3) Let F be an archimedian totally ordered field.
   (a) (10P) Let $a \in F$ and $a > 0$. Show that there is an integer such that $a < n < a + 2$.
   (b) (15P) Let $0 < a < b \in F$. Show that there exists an integer such that $a < \frac{n}{k} < b$. (Hint: First choose k such that $a + \frac{2}{k} < b$ and then try to apply (a) for $a' = \frac{k}{2}$?).
   (c) (10P) Show that $\mathbb{Q}$ is dense in F.

Solutions for Exam1

1): Let $A \subset \mathbb{R}$ be a closed bounded set. Show that

$$\sup A \in A \, .$$

Solution: Let us assume to the contrary that $\sup A \in A^c$. Since A^c is open, we find $\varepsilon > 0$ such that

$$(s - \varepsilon, s + \varepsilon) \subset A^c \, .$$

By the criterion for the supremum we find $a \in A$ such that

$$s - \varepsilon < a \, .$$

This contradiction proves the claim.

2): We consider our favorite metric space $(\mathbb{Z}, d_q)$ with the q-adic metric. For $x \in \mathbb{Z}$, we recall $r_k(x)$ to be the remainder with respect to q^k, i.e.

$$x \;=\; a_k q^k + r_k(x) \; , \; 0 \leq r_k(x) < q^k$$

for some $a_k \in \mathbb{Z}$. Show that the map $u : \mathbb{Z}_q \to \mathbb{R}$ (usual metric for $\mathbb{R}$) defined by

$$u(x) \; = \; \sum_{k=1}^{\infty} r_k(x) q^{-2k}$$

is Lipschitz.

Solution: Let $x, y \in \mathbb{Z}$ and n maximal such that $q^n | x - y$. Let $1 \leq k \leq r$ and

$$x \; = \; a_k q^k + r_k(x) \quad \text{and} \quad y \; = \; b_k q^k + r_k(x)$$

Then, we deduce from $k \leq n$ that

$$q^k | x - y \; = \; (a_k - b_k) q^k + (r_k(x) - r_k(y)) \; .$$

Hence q^k divides $(r_k(x) - r_k(y))$ which means $r_k(x) = r_k(y)$. From the geometric series, we deduce that

$$\begin{aligned} |u(x) - u(y)| &= |\sum_{k=n+1}^{\infty} (r_k(x) - r_k(y)) q^{-2k}| \\ &\leq \sum_{k=n+1}^{\infty} |r_k(x) - r_k(y)| q^{-2k} \\ &\leq \sum_{k=n+1}^{\infty} q^k q^{-2k} \; = \; q^{-(n+1)} \frac{1}{1 - \frac{1}{q}} \\ &= \frac{q^{-1}}{1 - \frac{1}{q}} d_q(x, y) \; . \end{aligned}$$

This shows that u is Lipschitz with constant $\frac{q^{-1}}{1-\frac{1}{q}}$. ∎

3): Let F be an totally ordered archimedian field.

a) Let $a \in F$ and $a > 0$. Show that there is an integer such that $a < n < a + 2$.

Solution: Since F is archimedian we known that $A = \{n \in \mathbb{N} : a < n\}$ is not empty. Let n_0 be the smallest element in A. Assume $n_0 \; \geq \; a + 2$, then $n_0 - 1 \; \geq \; a + 1 > 1$. This contradiction shows $a < n_0 < a + 2$.

c) Let $0 < a < b \in F$. Show that there exists an integer such that $a < \frac{n}{k} < b$.

Solution: Since F is archimedian we may find k such that $\frac{1}{k} \; < \; \frac{b-a}{2}$. We define $a = \frac{k}{2} a$ and find n such that

$$ka < n < ka + 2 \; .$$

Then, we get

$$a < \frac{n}{k} < a + \frac{2}{k} < b \; .$$

Here we go.

c) Show that $\mathbb{Q}$ is dense in F.

Solution: Let $x \in F$ and $\varepsilon > 0$. Let $m > -x$ and consider $a = x + m > 0$. According to b) we find $n, k \in \mathbb{N}$ such that

$$a < \frac{n}{k} < a + \varepsilon \, .$$

This implies

$$x < \frac{n - km}{k} < x + \varepsilon \, .$$

Therefore

$$|x - \frac{n - km}{k}| < \varepsilon \, .$$

The assertion is proved. ■

Practice problems for the second exam

(1) Let $C \subset X$ be a totally bounded subset of a metric space X. Show that for every $\varepsilon > 0$ there are $x_1, ..., x_n \in C$ such that

$$C \subset B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon) \ .$$

Deduce from this that there exists a countable subset $D \subset C$ such that $D \subset \overline{C}$.

(2) Show that a totally bounded, complete set (in a metric space) is compact. (Hint: use previous problem).

(3) (a) Let $f : [0, a] \to \mathbb{R}$ be continuous on $[0, a]$, diff'le on $(0, a)$ and $f(0) = 0$, $f'(x) \geq 0$ for all $0 < x < a$. Show that $f(x) \geq 0$.

(b) $f : [0, a] \to \mathbb{R}$ be continuous on $[0, a]$, diff'le on $(0, a)$ and $f(0) = 0$. We assume $f'(x) \leq x^n$. Show that $f(x) \leq \frac{x^n}{n}$. (Hint: $g(x) = \frac{x^n}{n} - f(x)$).

(c) Let $f : [0, a] \to \mathbb{R}$ continuous, differentiable, $f(0) = 0$ and $f'(x) = f(x)$. Let $c = \sup_{x \in [0,a]} |f(x)|$. Show by induction that

$$|f(x)| \leq \frac{c}{n} x^n \ .$$

Deduce from this that $f(x) = 0$ on $[0, a]$.

(d) Conclude that $f'(x) = f(x)$ with $f(0) = 1$ has at most one solution.

(4) Show that

$$\sum_n n x^n = \frac{2 - x}{(1 - x)^2}$$

holds for $-1 < x < 1$.

Exam2-347

Name:

An A will be awarded if you obtain 80 or more points (Problem 3) or 4)!)

(1) (15P) Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b) such that $|f'(t)| \leq 1$. Show that f is Lipschitz with constant 1.

(2) Let(X,d) be a metric space.

(a) (10P) Let $\varepsilon > 0$ and $(x_n) \subset X$ such that

$$d(x_n, x_m) \geq \varepsilon$$

for all $n \neq m$. Show that (x_n) does not have a convergent subsequence.

(b) (30P) Show that a sequentially compact set $C \subset X$ is totally bounded. (Hint: Proceed by contradiction and construct by induction a sequence $x_n \in C$ such that $x_{n+1} \notin B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)$.)

(3) (25P)

(a) Let (a_n) be a sequence such that $(\limsup_n a_n)^{-1} > r$ and such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies

$$f'''(x) = f(x) \quad \text{on} \quad (-r, r) \quad \text{and} \quad f(0) = f'(0) = 1\,,\ f''(0) = 2\,.$$

Show that $a_0 = a_1 = a_2 = 1$ and that

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)(n+1)}\,.$$

(Hint you may use the fact that two power series coincide if and only if the coefficients coincide, (the proof of this fact yields extra credit))

(b) Show that there exists a solution of $f'''(x) = f(x)$, $f(0) = 1 = f'(0)$ and $f''(0) = 2$ which is defined on $\mathbb{R}$.

(4) (25P) Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(0) = 0$, $g(y) > 0$ for all $y > 0$ and $g(y) < 0$ for all $y < 0$. Let f a solution of

$$f'(x) \;=\; g(f(x))$$

on $[0, \infty)$. Show that only the two alternatives may occur

(a) $f(x) \geq 0$ for all $x > 0$ or

(b) $f(x) \leq 0$ for all $x < 0$.

(Hint: Investigate the situation $0 < x_0$ with $f(x_0) > 0$ assume that there exists $x_1 > x_0$ with $f(x_1) \leq 0$. A clever choice of x_1 leads to a contradiction by the mean value theorem).

Exam2-347

(1) Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b) such that $|f'(t)| \leq 1$. Show that f is Lipschitz with constant 1.

Proof: Let $a \leq x \leq y \leq b$, we apply the MVT and find $x < z < y$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(z)$$

and hence

$$|f(y) - f(x)| = |f'(z)||x - y| \leq |x - y| .$$

■

(2) Let(X,d) be a metric space.

(a) Let $\varepsilon > 0$ and $(x_n) \subset X$ such that

$$d(x_n, x_m) \geq \varepsilon$$

for all $n \neq m$. Show that (x_n) does not have a convergent subsequence.

Proof: Let (n_k) be a subsequence and assume $(x_{n_k}))k$ is Cauchy then there exists an k_0 such that

$$d(x_{n_k}, x_{n_m}) < \varepsilon$$

for all $k, m > k_0$. For $k = k_0+1$, $m = k_0+2$ we obtain contradiction. ■

(b) Show that a sequentially compact set C is totally bounded. (Hint: Proceed by contradiction and construct by induction a sequence x_n such that $x_{n+1} \notin B(x_1, \varepsilon) \cup \cdots B(x_n, \varepsilon)$.)

Proof: Let $\varepsilon > 0$ if there are not $x_1, ..., x_n$ such that $C \subset B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)$. This means:

$$\forall_{n\in\mathbb{N}} \forall_{x_1,...,x_n \in C} \exists x \in C \setminus B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon) .$$

Since every empty set it totally bounded, we may find $x_1 \in C$. We apply our condition for $n = 1$ and find $x_2 \in C \setminus B(x_1, \varepsilon)$. Let us assume we have fund $x_1, ..., x_n$ such that

$$x_{k+1} \in C \setminus B(x_1, \varepsilon) \cup \cdots \cup B(x_k, \varepsilon) .$$

Then we apply the criterion for n and find $x \in C \setminus B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)$. We call this $x_{n+1} = x$. In this way (axiom of choice) we find (x_k) such that

$$x_{k+1} \in C \setminus B(x_1, \varepsilon) \cup \cdots \cup B(x_k, \varepsilon)$$

holds for all $k \in \mathbb{N}$. In particular

$$d(x_j, x_k) \geq \varepsilon$$

for all $j < k$. By symmetry of the distance the condition in a) is satisfied and hence (x_k) does not have a convergent subsequence. ■

(3) (25P)

(a) Let (a_n) be a sequence such that $(\limsup_n a_n)^{-1} > r$ and such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies

$$f'''(x) = f(x)$$

on $(-r, r)$ and

$$f(0) = f'(0) = f''(0) = 1 .$$

Show that $a_0 = a_1 = a_2 = 1$ and that

$$a_{n+3} = \frac{a_n}{(n+3)(n+2)} .$$

Proof: Applying the theorem on differentiation of power series several times, we find

$$f'''(x) = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n .$$

With the uniqueness of power series, we get

$$(n+3)(n+2)(n+1) a_{n+3} = a_n .$$

This yields the assertion. ■

Proof: We define $a_0 = a_1 = 1 = a_2$ and inductively

$$a_{3n+k} = \frac{a_{3(n-1)+k}}{(3n+k)(3n+k-1)(3n+k-2)} = \frac{a_{3(n-2)+k}}{(3n+k)(3n+k-1)(3n+k-2)(3n+k-3)(3n+}$$

$$= \frac{k!}{(3n+k)!} a_k$$

This yields

$$a_n = \begin{cases} \frac{1}{n!} & \text{if } n \cong 0 \mod 3 \text{ or } n \cong 1 \mod 3 \\ \frac{2}{n!} & \text{if } n \cong 2 \mod 3 \end{cases} .$$

Since $\lim_n (n!)^{\frac{1}{n}} = \infty$ we deduce $\lim_n a_n^{\frac{1}{n}} = \infty$ and therefore

$$f(x) = \sum_n a_n x^n$$

is 3-times differentiable. Using the theorem on differentiation of power series, we deduce $f'''(x) = f(x)$ and $f(0) = a_0 = 1$, $f'(0) = a_1 = 1$, $f''(0) = 2a_1 = 2$.

(4) (25P) Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(0) = 0$, $g(y) > 0$ for all $y > 0$ and $g(y) < 0$ for all $y < 0$. Let f a solution of

$$f'(x) = g(f(x))$$

on $[0, \infty)$. Show that only the three alternatives may occur

(a) $f(x) = 0$ for all $x = 0$,
(b) $f(x) > 0$ for all $x > 0$ or
(c) $f(x) < 0$ for all $x < 0$.

(Hint: Investigate the situation $0 < x_0$ with $f(x_0) > 0$ assume that there exists $x_1 > x_0$ with $f(x_1) \leq 0$. A clever choice of x_1 leads to a contradiction by the mean value theorem).

Proof: Assume $f(x_0) > 0$ and that there exists $x > x_0$ with $f(x_1) \leq 0$. Since f is continuous

$$x_1 = \inf\{x > x_0 : f(x) \leq 0\}$$

satisfies $x_1 > x_0$. Since $x_1 = \lim_n x_1 - \frac{1}{n}$ and $f(x_1 - \frac{1}{n}) > 0$ for all $n \in \mathbb{N}$, we see that

$$f(x_1) = \lim_n f(x_1 - \frac{1}{n}) \geq 0 .$$

By definition of the inf there exists $y_n > x_1$ such that $f(y_n) \leq 0$ and $f(y_n) \leq 0$. Thus by continuity of f, we find $f(x_1) = \lim_n f(y_n) \leq 0$. Therefore $f(x_1) = 0$. By the mean values theorem we find $x_0 < y < x_1$ such that

$$-\frac{f(x_1)}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(y) = g(f(y)) .$$

However, $f(y) > 0$ because $y < x_1$ and therefore $g(f(y)) > 0$ -a contradiction.

This means that if for some $x_0 > 0$ we have $f(x_0) > 0$, then for all $x > x_0$, we have $f(x) > 0$. Since we can apply this for $f' = -gf$ we deduce that there are no sign changes.

Now, given a solution, we consider two cases: Either for all x we have $f(x) \leq 0$ or there exists an x with $f(x) > 0$. In the second case, we have $f(y) > 0$ for all $y > x$. Then either $f(y) \geq 0$ for all $0 < y < x$ and we are done or there exists $0 < y < x$ with $f(y) < 0$. Using the first part for $f' = -g(f)$, we deduce $f(z) < 0$ for all $z > y$. In particular, $f(x) < 0$ a contradiction. This contradiction proves that $f(x) \geq 0$ for all $x > 0$.

(1) Let $g : \mathbb{R} \to \mathbb{R}$ be an strictly increasing function. Let f_1 and f_2 continuous on $[0,1]$ and differentiable on $(0,1)$ such that $f_1(0) = 1 = f_2(0)$ and

$$f_1'(x) \ = \ g(f_1(x)) \quad \text{and} \quad f_2'(x) \ = \ g(f_2(x)) \, .$$

(a) Assume that $f_1(x_0) > f_2(x_0)$. Show that for all $1 \leq x > x_0$

$$f_1(x) > f_2(x) \, .$$

(Hint: Proceeding by contradiction, we may consider $x_2 \ = \ \inf\{x : x > x_0 f_1(x) \leq f_2(x)\}$. By continuity $f_1(x_2) = f_1(x_2)$ and $x_2 > x_0$ (show this)-mean value theorem!)

Proof: We assume that there exists $x_0 < x \leq 1$ such that $f_1(x) \leq f_2(x)$. Then the set

$$A \ = \ \{x : x > x_0 f_1(x) \leq f_2(x)\}$$

is not empty and hence $x_2 \ = \ \inf A$ is well defined. Let $(y_n) \in A$ converge to x_2. Then

$$f_1(y_n) - f_2(y_n) \leq 0$$

for all n. Hence by continuity

$$f_1(x_2) - f_2(x_2) \ = \ \lim_n f_1(y_n) - f_2(y_n) \leq 0 \, .$$

In particular, we deduce $x_2 > x_0$. We consider $z_n \ = \ x_2 - \frac{1}{n}$. Then for $n > n_0$ we have $x_n \in A^c$ and thus $f_1(z_n) > f_2(z_n)$. Passing to the limit implies

$$f_1(x_2) \ \geq \ f_2(x_2) \, .$$

Now, we apply the MWT and find $x_0 < z < x_2$ such that

$$\begin{aligned} \frac{f_2(x_0) - f_1(x_0)}{x_2 - x_0} &= \frac{f_1(x_2) - f_2(x_2) - (f_1(x_0) - f_2(x_0))}{x_2 - x_0} \\ &= f_1'(z) - f_2'(z) \ = \ g(f_1(z)) - g(f_2(z)) > 0 \, . \end{aligned}$$

Since the left hand term is negative this is a contradiction. ■

(b) (EXTRA CREDIT) Conclude that either $f_1(x) > f_2(x)$ for all $0 < x < 1$ or $f_1(x) \leq f_2(x)$ for all $0 \leq x \leq 1$. (Hint: Consider $x_3 = \inf\{x : f_1(x) > f_2(x)\}$. If $x_3 = 0$ you are done (why). If $x_3 > 0$ you will find $f_1(x) = f_2(x)$ for all $0 < x < x_2$ and $f_1(x) > f_2(x)$ for all $x > x_2$.)

Solution: Either $f_1(x) \leq f_2(x)$ for all x or

$$A = \{x > 0 : f_1(x) > f_2(x)\}$$

is non-empty. If $\inf A = 0$, we consider $x > 0$ and find $0 < x_0 < 0$ such that $f_1(x_0) > f_2(x_0)$ and the a) implies $f_1(x) > f_2(x)$. If $\inf A = x_2 > 0$, the same argument as above shows that $f_1(x_3) = f_2(x_3)$. Hence, as a before, we find $f_1(x) > f_2(x)$ for all $x > x_3$. Now, we consider $0 < x < x_3$. If $f_1(x) < f_2(x)$ we then we see that $\{y > x : f_1(x) \geq f_2(x)\}$ is not empty because x_3 is such a point. Interchanging the roles of f_1 and f_2 in a) we deduce a $f_1(y) < f_2(y)$ for all $y > x$-a contradiction. Therefore, we find $f_1(x) = f_2(x)$ for all $0 \leq x \leq x_2$. In particular, $f_1(x) \geq f_2(x)$ for all $x \geq 0$. ■

Final exam 347-2004

Name:

(1) Give an example of subset $C \subset X$, (X, d) a metric space such that
   (a) (5P) C is closed, but not compact.
   (b) (5P) C is totally bounded, but not compact.
   (c) (5P) C is compact but not connected.
   (d) (10P) C is sequentially compact, but not compact.

(2) (30P) Let (a_n) be a sequence such that $\limsup_n |a_n|^{\frac{1}{n}} > 1$, $a_0 = 1$ and

$$\sum_{n=0} a_n \;=\; 3$$

Show that there exists $0 < x < 1$ such that

$$2 \;=\; \sum_{n} n a_n x^{n-1} \,.$$

Hint: You have to combine two theorems form the course.

(3) Show that the set

$$\{(x, y)\ : x \geq -1, -1 \leq y \leq 1\ e^x\ \leq\ \cos(x + y)\}$$

is compact. Justify your arguments by introducing some auxilary functions.

(4) (40P) Show Dini's theorem: Let (f_n) be a sequence of continuous functions on $[0,1]$ such that

$$f_n(x) \leq f_{n+1}(x)$$

and the limit function $g(x) = \lim_n g_n(x)$ is continuous. Then (f_n) converges uniformly.

(Hint: For every $x \in C$ find $n(x)$ such that $g(x) < f_{n(x)} + \frac{\varepsilon}{3}$. Then you can find $\delta(x) > 0$ such that $d(x,y) < \delta(x)$ implies $|g(y) - g(x)| < \frac{\varepsilon}{3}$ and $|f_{n(x)}(x) - f_{n(x)}(y)| < \frac{\varepsilon}{3}$ (why ?). Apply compactness and define $n_0 = \max\{n(x_1), ..., n(x_m)\}$. Use $g(y) \geq f_n(y) \geq f_{n(x_i)}$ (which i?) to conclude.)

(5) (40P) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ and a_0,....,$a_{n-1} \in \mathbb{R}$. Let f be defined by

$$f(x) \ = \ a_i$$

whenever $x_i \leq x < x_{i+1}$. Show that f is integrable and that

$$\int_0^1 f(x)dx \ = \ \sum_{i=0}^{n} a_i(x_{i+1} - x_i)\ .$$

(Hint: Choose $\delta > 0$ small enough and consider the partition I_δ with points $0 = x_0 < x_1 - \delta < x_1 < x_2 - \delta < x_2 < \cdots x_n - \delta < x_n$. Estimate $\overline{\Sigma}(f, I_\delta) - \underline{\Sigma}(f, I_\delta)$ and don't forget a picture.)

Final exam solution 347-2004

Name:

(1) Give an example of subset $C \subset X$, (X,d) a metric space such that
   (a) C is closed, but not compact. Indeed, $C = \mathbb{R}$ is closed but not compact.
   (b) C is totally bounded, but not compact. Indeed, $C = [0,1)$ is totally bounded, because $C \subset [0,1]$ and totally bounded subsets of totally bounded sets are totally bounded.
   (c) C is compact but not connected. Indeed, $C = [0,1] \cup [2,3]$ is closed and bounded, hence compact and $V = [0,1] = (-1,1.5) \cap C$ and $W = [2,3] = (1.5,4) \cap C$ are open, satisfy $V \cup W = C$ and are both not empty.
   (d) C is sequentially compact, but not compact. Such a set cannot exists (This is the final version of the characterization theorem).

(2) Show the mean value theorem for differentiable functions. State explicitly the assumptions. (You may use Rolle's Lemma).

Solution: MVT: f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $a < x < b$ such that

$$f'(x) = \frac{f(b)-f(a)}{b-a} .$$

Proof: We defined

$$g(x) = f(x) - f(a) - (x-a)\frac{f(b)-f(a)}{b-a} .$$

Then $g(a) = 0 = g(b)$. g is continuous on $[a,b]$ and differentiable on (a,b) and hence there exists $a < x < b$ such that

$$0 = g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} .$$

(3) (30P) Let (a_n) be a sequence such that $\limsup_n |a_n|^{\frac{1}{n}} > 1$, $a_0 = 1$ and

$$\sum_{n=0} a_n = 3$$

Show that there exists $0 < x < 1$ such that

$$2 = \sum_n n a_n x^{n-1} .$$

Hint: You have to combine two theorems form the course.

Proof: Define

$$f(x) = \sum_n a_n x^n$$

By the differentiation theorem for power series ($R > 1$ by assumption)

$$f'(x) = \sum_n n a_n x^{n-1} .$$

The mean value theorem for differentiation provides x such that

$$2 = \frac{f(1) - f(0)}{1} = f'(x) .$$

Indeed, $f(0) = a_0$ and $f(1) = \sum_n a_n = 3$.

(4) Show that the set

$$\{(x, y) : x \geq -1, -1 \leq y \leq 1\ e^x \leq \cos(x + y)\}$$

is compact. Justify your arguments.

Proof: We define $g_0(x, y) = e^x$, $g_1(x, y) = -\cos(x+y) = -\cos(h(x, y))$ where $h(x, y)$ is given by $h(x, y) = x + y$. We first note that the functions $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are obviously continuous. Then h is the sum of continuous functions and hence continuous. g_1 is the composition of a continuous function with a continuous function, hence continuous. g_0 is the composition of π_1 with the continuous function and hence continuous. Therefore the set

$$A = \{(x, y) : g_0(x, y) - g_1(x, y) \leq 0\} = \{(x, y) : g_0(x, y) - g_1(x, y) > 0\}^c$$

is the complement of an pen set because $g_0 - g_1$ is continuous. The set A' defined in the problem satisfies

$$A' = A \cap [-1, 1] \times [-1, 0]$$

because for $x > 0$ we have $e^x > 1 \geq \cos(x + y)$. ■

(5) Show Dini's theorem: Let C be a compact subset f a metric space and (f_n) be a sequence of continuous functions on $[0, 1]$ such that

$$f_n(x) \leq f_{n+1}(x)$$

and the limit function $g(x) = \lim_n g_n(x)$ is continuous. Then (f_n) converges uniformly.

Solution: Let $\varepsilon > 0$. For every $x \in C$ there exists $n(x)$ such that

$$g(x) - \frac{\varepsilon}{3} < f_{n(x)}(x) .$$

Since $f_{n(x)}$ is continuous there exists $\delta_1(x)$ such that

$$d(x, y) < \delta_1(x)$$

implies

$$|f_{n(x)}(x) - f_{n(x)}(y)| < \frac{\varepsilon}{3}$$

Since g is continuous, there exists $\delta_2(x) > 0$ such that

$$d(x,y) < \delta_2(x)$$

implies

$$|g(y) - g(x)| < \frac{\varepsilon}{3} .$$

let $\delta(x) \ = \ \min\{\delta_1(x), \delta_2(x)\}$. Then

$$C \subset \bigcup_{x \in C} B(x, \delta(x)) .$$

By compactness, there exist $x_1, ..., x_m$ such that

$$C \subset B(x_1, \delta(x_1)) \cup \cdots \cup B(x_m, \delta(x_m)) .$$

Let $n > \max\{n(x_1), ..., n(x_m)\}$. Let $y \in C$. Then there exists x_i such that $d(y, x_i) < \delta(x_i)$. Therefore, we get

$$|g(y) - g(x)| < \frac{\varepsilon}{3}$$

and hence

$$f_n(y) = f_{n(x)}(y) \ \geq \ f_{n(x)} - \frac{\varepsilon}{3} \ \geq \ g(x) - \frac{2\varepsilon}{3} \ \geq \ g(y) - \varepsilon .$$

This implies with $f_n(y) \leq g(y)$ that

$$|g(y) - f_n(y)| < \varepsilon$$

for all $y \in C$ and $n > \max\{n(x_1), ..., n(x_m)\}$. ■

(6) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ and a_0,....,$a_{n-1} \in \mathbb{R}$. Let f be defined by

$$f(x) \ = \ a_i$$

whenever $x_i \leq x < x_{i+1}$. Show that f is integrable and that

$$\int_0^1 f(x)dx \ = \ \sum_{i=0}^{n} a_i(x_{i+1} - x_i) .$$

Solution: Let $\varepsilon >$ and $\delta < \min_{i=0,...,n} x_{i+1} - x_{i+1}$. Let $I = \{x_1, ..., x_{n-1}$ and $J = \{x_1 - \delta, ..., x_{n-1} - \delta\}$. We consider $I_\delta = I \cup J$. We first observe that

$$\inf_{[x_i, x_{i+1} - \delta]} f \ = \ a_i \ = \ \sup_{[x_i, x_{i+1} - \delta]} f .$$

However,

$$\inf_{[x_{i+1}-\delta],x_{i+1}} f \;=\; \min\{a_i, a_{i+1}\}$$

and

$$\sup_{[x_{i+1}-\delta],x_{i+1}} f \;=\; \max\{a_i, a_{i+1}\}\;.$$

Therefore, we get

$$\begin{aligned}\overline{\Sigma}(f, I_\delta) - \underline{\Sigma}(f, I_\delta) &= \sum_{i=0}^{n-1} \delta(\max\{a_i, a_{i+1}\} - \min\{a_i, a_{i+1}\}) \\ &\leq \delta n(\max_{i=0,\ldots,n} a_i - \min_{i=0,\ldots,n} a_i)\end{aligned}$$

Hence $\delta < \frac{\varepsilon}{n(\max_{i=0,\ldots,n} a_i - \min_{i=0,\ldots,n} a_i)}$ provides

$$\overline{\Sigma}(f, I_\delta) - \underline{\Sigma}(f, I_\delta) < \varepsilon\;.$$

In particular, we get

$$\underline{\Sigma}(f, I_\delta) \leq \int_0^1 f \leq \overline{\Sigma}(f, I_\delta)\;.$$

The same estimate as above yields

$$|\sum_{i=1}^{n} a_i(x_{i+1} - x_i) - \underline{\Sigma}(f, I_\delta)| \;\leq\; \delta n(\max_{i=0,\ldots,n} a_i - \min_{i=0,\ldots,n} a_i)$$

The assertion follows. ■