CHAPTER 1

Metric spaces

1. Introduction

Definition 1.1. *A metric space* (X, d) *is given by a set* X *and* $d : X \times X \to [0, \infty]$ *satisfying*

(1) $d(x, y) = d(y, x)$,
(2) $d(x, y) = 0 \Leftrightarrow x = 0$,
(3) $d(x, y) \le d(x, z) + d(z, y)$

for all $x, y, z \in X$.

Examples: Discrete metric, $\mathbb{R}$, $\mathbb{R}^n$,

Lemma 1.2. *(Cauchy-Schwarz)* $|(x, y)| \le |x||y|$

Lemma 1.3. $|x + y| \le |x| + |y|$.

Examples: max-distance, sum distance for codes. Save code transmission.

Example: Chicago suburb metric.

Example: p-adic metric on $\mathbb{Q}$.

Definition 1.4. *A map* $T : (X, d) \to (X', d')$ *is called Lipschitz if there exists a constant* $C > 0$ *with*

$$d(T(x), T(y)) \le Cd(x, y)$$

for all $x, y \in X$.

Definition 1.5. *Let* $S \subset X$ *by subset, then* (S, d) *defined by* $d|_{S\times S}$ *is a called a subspace.*

Proposition 1.6. *Let* $S \subset X$ *be a finite set. Then the mapping* $id : S \to S$ *induces Lipschitz map* $id : (S, d) \to (S, d_{disc})$ *and a Lipschitz map* $id : (S, d_{disc}) \to (S, d)$

Examples: Lipschitz maps on real line.

2. Topology in metric spaces

In the following (X, d) is a fixed metric space.

Definition 2.1. *1)* $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$. *2)* $\bar{B}(x, \varepsilon) = \{y : d(x, y) \le \varepsilon\}$.

Problem: Describe open and closed for the Chicago suburb metric.

Definition 2.2. *$O \subset X$ is called open, if for all $x \in O$, there exists $k \in \mathbb{N}$ such that*

$$B(x, \frac{1}{k}) \subset O \, .$$

A set C is closed if C^c is open.

Warning: O not open does not imply O closed. (find examples)

Lemma 2.3. *$\delta = d(x,y)$. Then*

$$B(y, \delta) \subset B(x, \varepsilon) \, .$$

Corollary 2.4. *An open ball $B(x, \varepsilon)$ is open. A closed ball is closed.*

Example: $\{(x,y) \ : \ x < y^2\}$ is open.
Now, we want to define the closure of a set.

Definition 2.5. *Let B be a set. Then*

$$\bar{B} \ = \ \{x \in X \ : \ \forall \varepsilon > 0 B(x, \varepsilon) \cap B \neq \emptyset\}$$

Proposition 2.6. *$\bar{B}$ is the smallest closed set which contains B.*

Definition 2.7. *(x_n) converges to x if*

$$\forall \varepsilon > 0 : \exists n_0 \forall n > n_0 \; d(x_n, x) < \varepsilon \, .$$

Notation:

$$\lim_n x_n \ = \ x$$

(x_n) is called convergent if there exists $x \in X$ with $\lim_n x_n = x$.

Example: $x_n = (\cos 1/n, \sin 1/n)$ with different metrics.
Project: $(x_n) \subset \mathbb{R}^d$ converges if all the component converge.

Proposition 2.8.

$$\bar{B} \ = \ \{x \ : \ \exists (x_n) \subset B \; \lim_n x_n = x\} \, .$$

Definition 2.9.

$$\overset{\circ}{B} = \{x \in B \ : \ \exists \varepsilon > 0 : B(x, \varepsilon) \subset B\}$$

Proposition 2.10. *Let B be a set. Then*

$$X \ = \overset{\circ}{B} \cup \partial B \cup \overset{\circ}{B^c}$$

is a partition of X. ∂B is the boundary.

3. Continuous functions

A map $T : (X, d) \to (Y, d')$ is called continuous at x if $\lim_n x_n = x$ implies

$$\lim_n T(x_n) \;=\; T(x)\;.$$

We use $C_x(X, Y)$ for the set of map $T : X \to Y$ which are continuous at x.
Examples: $T(x) = |x|$.

PROPOSITION 3.1. *$C_x(X, \mathbb{R})$ is closed under pointwise addition, scalar multiplication and pointwise multiplication.*

PROPOSITION 3.2. *$T : X \to Y$. TFAE*

(1) *T is continuous at x.*
(2) *$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y : (d(x, y) < \delta \Rightarrow d(T(x), T(y)) < \varepsilon.)$*

Project: Find a non-continuous function such that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall y : (d(x, y) < \varepsilon \Rightarrow d(T(x), T(y)) < \delta)\;.$$

DEFINITION 3.3. *$T : X \to Y$ is continuous if T is continuous at every point.*

PROPOSITION 3.4. *Polynomials are continuous.*

PROPOSITION 3.5. *T is continuous if and only if $T^{-1}(O)$ is open for every open set $O \subset Y$.*

Project: How to show that a map is not continuous (for example identity map with two different metrics on $\mathbb{R}^2$)?
Project for extra credit: Connected topological spaces.

4. Complete metric spaces

DEFINITION 4.1. *A sequence (x_n) is called Cauchy sequence (short (Chy) if*

$$\forall \varepsilon > 0 \exists n_0 \forall n, m > n_0\; d(x_n, x_m) < \varepsilon\;.$$

(Suffices for all $\varepsilon = \frac{1}{k}$).
Example: (p^n) is Chy in p-adic metric.

LEMMA 4.2. *(x_n) Chy and (x_{n_k}) convergent, then (x_n) convergent.*

DEFINITION 4.3. *(X, d) is called complete of every Cauchy sequence is convergent.*

Examples: Finite sets are complete.
Examples: $\mathbb{Q}$ is not complete, $(0, 1]$ is complete.

REMARK 4.4. *a) For completeness it suffices to check the convergece of sequence with* $d(x_n, x_{n+1}) < 2^{-n}$.
b) If (X, d) *is given by a norm*

$$d(x, y) \ = \ \|x - y\| \ ,$$

then (X, d) *is complete if and only if every absolutely convergent sequence is converging.*

PROPOSITION 4.5. *Closed subsets of complete metric spaces are complete.*

REMARK 4.6. *Every metric space has a completion. (Note here, hard enough for reals).*