1. The real numbers

Definition 1.1. *A group* $(G, \cdot, e)$ *is given by a set* G*, a map* $\cdot : G \times G \to G$ *and* $e \in G$ *such that*

(1) $\cdot(f, \cdot(g, h)) \;=\; \cdot(\cdot(f, g), h)$
(2) $\cdot(f, e) = \cdot(e, f) = f$
(3) *There exists* k *such that* $\cdot(f, k) = e = \cdot(k, f)$

for all $f, g, h \in G$*. Notation* $gh = \cdot(g, h)$*.* G *is called commutative if in addition*

$$gh \;=\; hg$$

for all $g, h \in G$*.*

Definition 1.2. *A field is given by* $(F, +, 0, \cdot, 1)$ *such that* $(F, +, 0)$ *is a commutative group,* $(F \setminus \{0\}, \cdot, 1)$ *is a commutative group and*

$$x(y + z) \;=\; xy + xz$$

for all $x, y, z \in F$*.*

Project: Show that a filed has at least two elements.

Definition 1.3. $(X, \leq)$ *is pre-ordered if* $\leq \subset X \times X$ *satisfies*

(1) $(x, y) \in \leq \Leftrightarrow (y, x) \in \leq$
(2) $(x, x) \in \leq$.
(3) $(x, y) \in \leq$, $(y, z) \in \leq$ *implies* $(x, z) \in \leq$.

for all $x, y, z \in X$*. Notation* $x \leq y$ *instead of* $(x, y) \in \leq$*. If moreover*

$$(x, y) \in X \times X \Rightarrow x \leq y \wedge y \leq x \;,$$

then X *is called totally ordered.*

Definition 1.4. *An ordered field is a field with a total order* $\leq$ *such that* $x \leq y$ *implies*

$$x + z \leq y + z$$

and if moreover $z \geq 0$

$$xz \leq yz$$

for all $x, y, z \in F$*.*

Project: $\mathbb{Q}$ is an ordered field.

Lemma 1.5. *Every ordered field contains a copy of* $\mathbb{Q}$*.*

DEFINITION 1.6. $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ *and* $d(x,y) = |x-y|$.

DEFINITION 1.7. *Let F be an ordered field. If $d : X \times X \to F$ satisfies the axioms of a metric space, we call it an F-metric space. In the definition of convergence and completeness, we replace $\varepsilon > 0$ by $\frac{1}{k} > 0$.*

LEMMA 1.8. *$(\mathbb{Q}, d)$ is a $\mathbb{Q}$-metric space.*

THEOREM 1.9. *There exists an ordered field Y and a map $d : Y \times Y \to Y$ such that such that (Y, d) is a complete Y-metric space and $\mathbb{Q}$ is dense, i.e. for every $x \in Y$ and $k \in \mathbb{N}$, there exists $q \in Q$ with*

$$d(x,q) < \frac{1}{k} \, .$$

REMARK 1.10. *Such an object is unique up to isomrophism*

PROPOSITION 1.11. *(Properties)*

(1) $\lim \frac{1}{k} = 0$,
(2) *$a < b$ iff there exists $k \in \mathbb{N}$ with $a + \frac{1}{k} \leq b$*
(3) *if $a < b$, then there exists $q \in \mathbb{Q}$ such that*

$$a < q < b \, .$$

(4) *$a > 0$ and $\varepsilon > 0$, then there exists $k \in \mathbb{N}$ with $a < k\varepsilon$.*

DEFINITION 1.12. *A set $A \subset \mathbb{R}$ is bounded above if there exists $b \in \mathbb{R}$ such that*

$$a \leq b$$

for all $a \in A$.

THEOREM 1.13. *Let A be bounded from above. Then there exists $s \in \mathbb{R}$ such that*

(1) *s is an upper bound.*
(2) *If b is an upper bound for A, then $s \leq b$.*

$s = \sup A$ is called the supremum (=the least upper bound).

Project: Characterization for supremum.

COROLLARY 1.14. *Closed intervals are closed and complete.*