1. Integration

Definition 1.1. *A partition P of an interval $[a, b]$ is a finite subset of $[a, b]$ such that $a \in P$ and $b \in P$.*

$$\delta(P) \;=\; \max_{x\in P} \inf_{y\neq x} |x - y| \;.$$

If P_1 and P_2 are partitions, then $P_1 \cup P_2$ is a partition and

$$\delta(P_1 \cup P_2) \;\leq\; \min\{\delta(P_1), \delta(P_2)\} \;.$$

Remark 1.2. *If P is a parition, we may define*

$$x_0 \;=\; a$$

and inductively

$$x_{k+1} \;=\; \inf\{x \in P \;:\; x > x_k\} \;.$$

Since P is finite we find $m \in \mathbb{N}$ such that $x_m = b$. Thus

$$x_0 < x_1 < \cdots < x_m \;=\; b \;.$$

We will use the notation $P = \{a = x_0 < x_1 < .... < x_m = b\}$.

Let $f : [a, b] \to \mathbb{R}$ be a function and $P = \{a = x_0 < x_1 < .... < x_m = b\}$ be a partition. Then we define

$$M_i \;=\; \sup_{x_i\leq x\leq x_{i+1}} f(x) \quad , \quad m_i \;=\; \sup_{x_i\leq x\leq x} f(x)$$

and

$$M(f, P) \;=\; \sum_{i=0}^{m-1} m_i(x_{i+1} - x_i) \quad , \quad m(f, P) \;=\; \sum_{i=0}^{m-1} m_i(x_{i+1} - x_i) \;.$$

Lemma 1.3. *Let P_1, P_2 be two partitions.*

(1) $M(f, P_1) \;\geq\; M(f, P_1 \cup P_2)$.
(2) $m(f, P_1) \;\leq\; m(f, P_1 \cup P_2)$.
(3) $m(f, P_1) \;\leq\; M(f, P_2)$.

(Proof by induction on $|P_1 \cup P_2|$.)

Definition 1.4. *A bounded function f is called D-integrable if*

$$\sup_P m(f, P) \;=\; \inf_P M(f, P) \;.$$

In this case

$$\int f \;=\; \inf_P M(f, P) \;.$$

Examples: $f(x) = x$, $f(x) = x^2$.

PROPOSITION 1.5. *If f and g are integrable and $\lambda \in \mathbb{R}$, then $f + \lambda g$ is integrable*

$$\int f + \lambda g \;=\; \int f + \lambda \int g \,.$$

Example: Exact integration formulas for polynomials of degree 2.

Example: Calculate $\sum\limits_{k=1}^{n} \frac{1}{k}$!

THEOREM 1.6. *Continuous functions are integrable.*

LEMMA 1.7. *A continuous function on a compact set is uniformly continuous.*

THEOREM 1.8. *(Funadmental I) g continuous differentiable on (a, b) then g' is integrable and*

$$\int\limits_a^b g' \;=\; g(b) - g(a)\,.$$

THEOREM 1.9. *(Fundamental II) f integrable then*

$$F(x) \;=\; \int\limits_a^x f$$

is continuous. If f is continuous at x, then $F'(x) = f(x)$.

Applications: Integrartion by parts and change of variable.

COROLLARY 1.10. $R = (\limsup_n |a_n|^{\frac{1}{n}})^{-1}$. *Then*

$$\int\limits_0^x \sum_k a_k x^k \;=\; \sum_k a_k \frac{x^{k+1}}{k+1}\,.$$

Applications: Power series solution of certain differential equations.