

1. Suprema and Limsup

1.1. Supremum. In the following F is a totally ordered field. Let $A \subset F$, we say that b is an *upper bound* for A (short $A \leq b$) if

$$a \leq b$$

for all $a \in A$. We say that A is *bounded above* if there exists $b \in F$ with $A \leq b$. We say that $s = \sup A$ if $A \leq s$ and for all $A \leq s'$, we have $s \leq s'$. The supremum is *the least upper bound*.

Similarly, we use $b \leq A$ and $\inf A = -\sup -A$. A set A is *bounded* if there exists $b_1, b_2 \in \mathcal{F}$ such that

$$b_1 \leq A \leq b_2 .$$

Remark: If F is complete, then F is archimedean.

PROPOSITION 1.1. *If F is a complete, totally ordered field, then every nonempty set which is bounded above has a supremum.*

PROOF. We define $B = \{b \in F : A \leq b\}$.

Case 1): $\forall \varepsilon > 0 \exists a \in A, b \in B : b - a < \varepsilon$

We apply this to $\varepsilon = \frac{1}{k}$ and find $a_k \in A, b_k \in B$ such that $b_k - a_k < \frac{1}{k}$. Let us denote (AC) by $(a_k), (b_k)$ the sequences so obtained. Define

$$A_k = \max\{a_1, \dots, a_k\} \quad B_k = \min\{b_1, \dots, b_k\} .$$

Then, we have for all k

$$a_k \leq A_k \leq B_k \leq b_k \leq b_k + \frac{1}{k} .$$

This implies for $k \leq n$

$$A_k \leq A_n \leq B_k \leq a_k \leq A_k + \frac{1}{n}$$

and

$$a_k \leq B_n \leq B_k \leq b_k \leq b_k + \frac{1}{k} .$$

This implies

$$|A_k - A_n| \leq \frac{1}{k} \quad \text{and} \quad |B_k - B_n| \leq \frac{1}{k}$$

for all $k \leq n$. Therefore (A_k) and (B_k) are Cauchy sequences with $|A_k - B_k| \leq \frac{1}{k}$. Since F is complete, we see that $s = \lim_k A_k$ exists and satisfies

$$\lim_k A_k = s = \lim_k B_k .$$

Let us show that $s \in B$. Let $a \in A$ and $\varepsilon > 0$. Then there exists a k such that

$$b_k \leq s + \varepsilon$$

This implies

$$a \leq b_k \leq s + \varepsilon.$$

Since s is arbitrary, we find $a \leq s$. Now, let $s' \geq A$ and $\varepsilon > 0$. We find a_k such that

$$s \leq a_k + \varepsilon.$$

This implies

$$s \leq a_k + \varepsilon \leq s' + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce $s = \sup A$.

Case 2): $\exists \varepsilon > 0 \forall a \in A, b \in B b - a \geq \varepsilon$.

Since A and B are nonempty, we may fix $a_0 \in A$ and $b_0 \in B$. We show by induction that $b_j \in B$. This is true for $j = 0$. If the assertion is true for j , then our assumption implies

$$a \leq b_j - \varepsilon$$

for all $a \in A$. Thus $b_{j+1} \in B$. Choose k such that $k > \frac{(b_0 - a_0)}{\varepsilon}$. Then, we deduce

$$a_0 \leq b_k = b_0 - k\varepsilon < b_0 - (b_0 - a_0) < a_0.$$

This contradiction shows that there is no gap in archimedian fields and the assertion is proved. ■

PROPOSITION 1.2. *Let A be bounded above and $x \in F$. Then $x = \sup A$ if and only if*

- i) $A \leq x$,
- ii) $\forall \varepsilon > 0 \exists a \in A : x - \varepsilon < a$.

PROOF. It suffices to show that ii) is equivalent to

$$A \leq s' \Rightarrow x \leq s'$$

In the proof above we have seen that condition ii) implies that for every $s' \geq A$, we have $x \leq s'$.

If ii) is violated, we find $\varepsilon > 0$ such that $x - \varepsilon \geq A$. Thus $x - \varepsilon$ is an upper bound and $x - \varepsilon < x$. Thus x is not a lower bound for the upper bounds of A .

Thus ii) is equivalent to x is a lower bound for the upper bounds and we are done. ■

REMARK 1.3. *Every totally ordered field contains \mathbb{Q} . A Dedekind cut is a pair (A, B) of disjoint subsets of \mathbb{Q} such that*

$$A \cup B = \mathbb{Q}$$

and

$$a \leq b$$

for all $a \in A, b \in B$. If F is complete, we may define

$$x(A, B) = \sup A \in F.$$

It is easily shown that \mathbb{Q} is also dense in F and hence

$$F = \{x(A, B) : (A, B) \text{ Dedekind cut}\}.$$

Form this it is ‘easily’ that all complete totally ordered fields are isomorphic.

Therefore we will use \mathbb{R} for the unique totally ordered field.

1.2. Limsup. A sequence (x_n) is called bounded if $\{x_n : n \in \mathbb{N}\}$ is bounded. The limes superior is defined by

$$\limsup_n = \inf\{\sup\{x_n : n \geq m\} : m \in \mathbb{N}\}.$$

Sometimes we also write

$$\limsup_n = \inf_m \sup_{n \geq m} x_n.$$

PROPOSITION 1.4. *Let (x_n) be bounded and $x \in \mathbb{R}$. Then $x = \limsup_n x_n$ if and only if*

- i) $\forall \varepsilon > 0 \exists m \forall n \geq m : x_n \leq x + \varepsilon$.
- ii) $\forall \varepsilon > 0 \forall m \exists n \geq m : x_n > x - \varepsilon$.

PROOF. We introduce

$$y_m = \sup_{n \geq m} x_n.$$

Let us show that ii) is equivalent to $\inf y_m \geq x$.

Indeed, if $\inf_m y_m \geq x$, then $y_m \geq x$ for all $m \in \mathbb{N}$. Given $\varepsilon > 0$, we may then find $n \geq m$ such that

$$x_n > y_m - \varepsilon > x - \varepsilon.$$

Conversely, assume $\inf_m y_m < x$. Define $\delta = \frac{x - \inf_m y_m}{2}$. Then $x - \delta$ is not a lower bound for $\{y_m : m \in \mathbb{N}\}$ and hence there exists m such that $y_m < x - \delta$. By the definition of the supremum we get

$$x_n \leq x - \delta$$

for all $n \geq m$. This contradicts ii).

Now, we show that i) is equivalent to $\inf_m y_m \leq x$.

Indeed, $\inf_m y_m \leq x$ means that for every $\varepsilon > 0$, the real number $x + \varepsilon$ is not a lower bound. Hence, there exists $m \in \mathbb{N}$ such that $y_m < x + \varepsilon$. Thus for all $n \geq m$

$$x_n \leq y_m < x + \varepsilon .$$

Thus condition i) is satisfied.

Conversely, we assume $\inf_m y_m > x$ and define $2\varepsilon = \inf_m y_m - x$. Then $y_m > x + \varepsilon$ for all $m \in \mathbb{N}$. And hence there exists $n \geq m$ such that $x_n > x + \varepsilon$ for all $n \geq m$.

This means condition i) is violated.

The proof is complete. ■

PROPOSITION 1.5. *Let (x_n) be positive numbers*

$$\limsup_n x_n^{\frac{1}{n}} \leq \limsup_n \frac{x_{n+1}}{x_n} .$$

PROOF. we may assume (by convention) that $y = \limsup_n \frac{x_n}{x_{n+1}}$ is finite. Let $\varepsilon > 0$ and $z = y + \frac{\varepsilon}{2}$. Then there exists an n_0 such that $x_{n+1}/x_n \leq z$ for all $n \geq n_0$. This yields

$$\frac{x_n}{x_{n_0}} = \prod_{k=n_0}^{n-1} \frac{x_{k+1}}{x_k} \leq z^{n-n_0-1} .$$

Therefore, we find

$$x_n \leq z^n z^{-n_0-1} x_{n_0} .$$

This implies

$$x_n^{\frac{1}{n}} \leq z \left(\frac{x_{n_0}}{z^{n_0+1}} \right)^{\frac{1}{n}} .$$

Using $\lim_n a^{\frac{1}{n}} = 1$, we find $n_1 > n_0$ such that

$$\left(\frac{x_{n_0}}{z^{n_0+1}} \right)^{\frac{1}{n}} < \frac{y + \varepsilon}{y + \frac{\varepsilon}{2}}$$

for all $n \geq n_1$. This implies

$$x_n^{\frac{1}{n}} < \left(y + \frac{\varepsilon}{2} \right) \frac{y + \varepsilon}{y + \frac{\varepsilon}{2}} = y + \varepsilon$$

for all $n \geq n_1$. Since $\varepsilon > 0$ is arbitrary the assertion follows. ■

1.3. Limit sets of sequences. Let (x_n) be a sequence in a metric space (X, d) . We define

$$\text{Lim}(x_n) = \{x \in X : \exists(n_k) : x = \lim_k x_{n_k}\}.$$

Here, we allow only strictly increasing sequences of natural numbers.

PROPOSITION 1.6. $\text{Lim}(x_n) = \bigcup_m \overline{\{x_n : n \geq m\}}$. In particular, $x \in \text{Lim}(x_n)$ if and only if

$$\forall \varepsilon > 0 \forall m \in \mathbb{N} \exists n \geq m : d(x_n, x) < \varepsilon.$$

PROOF. The in particular part is direct application for the description of $\overline{A} = \{x : \forall \varepsilon > 0 : B(x, \varepsilon) \cap A \neq \emptyset\}$. Let us assume that $x = \lim_k x_{n_k}$ and $\varepsilon > 0$. Then there exists k_0 such that

$$d(x, x_{n_k}) < \varepsilon$$

for all $k \geq k_0$. Let $m \in \mathbb{N}$. Since $n_k \geq k$, we deduce that for $k = \max\{k_0 + 1, m\}$ we have $n_k \geq m$ and $k > k_0$ and hence

$$d(x, x_{n_k}) < \varepsilon.$$

Conversely, we assume that

$$\forall \varepsilon > 0 \forall m \in \mathbb{N} \exists n \geq m : d(x_n, x) < \varepsilon.$$

We apply this for $\varepsilon = 1$ and $m = 1$ and find n_1 such that

$$d(x, x_{n_1}) < 1.$$

We reapply the assumption to $\varepsilon = \frac{1}{2}$ and $m = n_1 + 1$ and find $n_2 > n_1$ such that

$$d(x, x_{n_2}) < \frac{1}{2}.$$

Inductively, we find $n_{k+1} > n_k$ such that

$$d(x, x_{n_{k+1}}) < \frac{1}{k+1}.$$

Therefore $x = \lim_k x_{n_k} \in \text{Lim}(x_n)$. ■

COROLLARY 1.7. $\text{Lim}(x_n)$ is closed.

let recall the notation $\max A = s$ of $s = \sup A$ and $s \in A$.

PROPOSITION 1.8. Let (x_n) be a bounded sequence. Then

$$\max \text{Lim}(x_n) = \limsup_n x_n.$$

PROOF. First, we show that $x = \limsup_n x_n \in \text{Lim}(x_n)$. Indeed, let $\varepsilon > 0$ and $m_0 \in \mathbb{N}$. Then there exists an $m_1 \in \mathbb{N}$ such that

$$x_n < x + \varepsilon$$

for all $n \geq m$ and in particular for all $n \geq \max\{m_0, m_1\}$. Using ii), we find $x > \max\{m_0, m_1\}$ such that also

$$x_n > x - \varepsilon.$$

Thus $d(x, x_n) < \varepsilon$. Therefore

$$\limsup_n x_n \in \text{Lim}(x_n)$$

and in particular, $\limsup_n x_n \leq \sup \text{Lim}(x_n)$. Now, let $y = \lim_k x_{n_k}$ and $\varepsilon > 0$. Then there exists k_0 such that

$$y < x_{n_k} + \frac{\varepsilon}{2}$$

for all $k \geq k_0$. We choose m such that

$$x_n < \limsup_n x_n + \frac{\varepsilon}{2}$$

for all $n \geq m$. Then, we may find $k \geq k_0$ such that $n_k \geq m$ and thus

$$y < x_{n_k} + \frac{\varepsilon}{2} < \limsup_n x_n + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce $y \leq \limsup_n x_n$. ■

COROLLARY 1.9. *Every bounded sequence in \mathbb{R} has a convergent subsequence.*

PROOF. $\limsup_n x_n \in \text{Lim}(x_n)$. ■

COROLLARY 1.10. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then there exists a $a \leq x_0 \leq b$ such that*

$$f(x_0) = \sup\{f(x) : a \leq x \leq b\}.$$

PROOF. Let us first assume that

$$\{f(x) : a \leq x \leq b\}$$

is bounded. Define $y = \sup\{f(x) : a \leq x \leq b\}$. Then, we find $x_n \in [a, b]$ such that

$$y - \frac{1}{n} \leq f(x_n) = y.$$

Let (n_k) such that $x_0 = \lim_k x_{n_k}$ exists. Then continuity implies

$$y = \lim_k f(x_{n_k}) = \lim_k f(x_{n_k}) = f(x_0).$$

Now, we have to show that $\{f(x) : a \leq x \leq b\}$ is bounded. If that is not the case, then we find $(x_n) \subset [a, b]$ such that $f(x_n) \geq n$. Let (x_{n_k}) a convergent subsequence and $y = f(x_0)$. Choose $n > y + 1$. By continuity, there exists k_0 such that

$$n_k \leq f(x_{n_k}) < y + 1 < n$$

for all $k \geq k_0$. Let $k \geq \max\{n, k_0\}$. Then

$$n \leq n_n < n$$

yields a contradiction. ■

1.4. series. We say $\sum_n x_n$ converges if the sequence of partial sums (s_n) defined by $s_n = \sum_{k=1}^n x_k$ is convergent. We say that $\sum_n x_n$ is *absolutely* convergent if $\sum_n |x_n|$ is convergent.

REMARK 1.11. *If $\sum_n x_n$ is absolutely convergent, then $\sum_n x_n$ is convergent. The converse is not true: $\sum_n \frac{(-1)^n}{n}$ is convergent, but $\sum_n \frac{1}{n}$ is not.*

PROPOSITION 1.12. *If $\limsup_n |x_n|^{\frac{1}{n}} < 1$, then*

$$\sum_n x_n$$

is absolutely convergent.

LEMMA 1.13. *Let $|a| < 1$. Then*

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

PROOF. We have

$$(1 + a + \cdots + a^{n-1})(1 - a) = 1 - a^n.$$

Therefore,

$$\sum_{k=1}^{n-1} a^k = \frac{1 - a^n}{1 - a}.$$

Since $|a| < 1$, we know that $\lim a^n = 0$ and the assertion follows. ■

PROOF. Let $\limsup_n |x_n|^{\frac{1}{n}} < a < 1$ (For example $a = \frac{1 + \limsup_n |x_n|^{\frac{1}{n}}}{2}$ will do.) Let n_0 such that

$$|x_n|^{\frac{1}{n}} \leq a$$

for all $n \geq n_0$. This implies

$$|x_n| \leq a^n$$

for all $n \geq n_0$. Let $\varepsilon > 0$ and r_0 such that $\frac{a^r}{1-a} < \varepsilon$ for all $r \geq r_0$. Let $m \geq n \geq \max\{n_0, r_0 + 1\}$. Then we get

$$\sum_{k=n+1}^m |x_k| \leq \sum_{k=n+1}^m a^k \leq \frac{a^{k+1}}{1-a} < \varepsilon.$$

The assertion is proved. ■

COROLLARY 1.14. *If $\limsup_n \frac{|x_{n+1}|}{|x_n|} < 1$, then*

$$\sum_n x_n$$

is absolutely convergent.

PROOF. $\limsup_n |x_n|^{\frac{1}{n}} \leq \limsup_n \frac{|x_{n+1}|}{|x_n|}$. ■