2. Hour exam-Math 361

Name:

1. (10P) Let X be a continuous random variable with density function f_X. Let $\alpha > 0$. Show that
$$f_{\alpha X}(x) \;=\; \frac{1}{\alpha} f_X(\frac{x}{\alpha})\,.$$
Solution: We have
$$F_{\alpha X}(a) \;=\; P(\alpha X \le a) \;=\; P(X \le a/\alpha) \;=\; \int_{-\infty}^{a/\alpha} f_X(x)dx \;=\; \int_{-\infty}^{a} f_X(x/\alpha)\frac{dx}{\alpha}\,.$$
Differentiation yields
$$f_{\alpha X}(a) \;=\; \frac{\partial}{\partial a} F_{\alpha X}(a) \;=\; \frac{f_X(a/\alpha)}{\alpha}\,.$$

2. (15P) Let U be uniformly distributed over $[0,1]$ and X be uniformly distributed over $[0,U]$. Calculate the distribution of U over X.

Solution: By definition $f_U(u) = 1$ for $0 \le u \le 1$ and 0 else. Moreover, by assumption
$$f_{X|U}(x|u) \;=\; \begin{cases} \frac{1}{u} & \text{if } 0 \le x \le u \\ 0 & \text{else} \end{cases}\,.$$
Therefore
$$f_{X,U}(x,u) \;=\; \frac{1}{u}$$
whenever $0 \le x \le u$. This yields
$$f_{U|X}(u|x) \;=\; \frac{f_{X,U}(x,u)}{f_X(x)}\,.$$
However, for $0 < x < 1$ we have
$$f_X(x) \;=\; \int_x^1 \frac{1}{u} du \;=\; \ln 1/x > 0\,.$$
This yields
$$f_{U|X}(u|x) \;=\; \frac{1}{-u\ln x}\,.$$
In other words U has the '$1/u$-distribution' over $[X,1]$ and X is itself $\ln(1/x)$-distributed (i.e. $f_X(x) = -\ln x$) on $[0,1]$.

3. (20P) Let X be exponential distributed with parameter 1 and Y be exponentially distributed with parameter X. Calculate

$$P(X < 1|Y = 1) \;=\; \int_0^1 f_{X|Y}(x|1)dx\,. \tag{0.0.1}$$

Solution: In general if f_Z is the density function of random variable Z, then

$$P(Z \in A) \;=\; \int\limits_A f_Z(x)dx\,.$$

(0.0.1) is the special case for the random variable $Z = E(X|Y = 1)$. In order to determine this distribution, we first determine the joint distribution function

$$f_{X,Y}(x,y) \;=\; f_{Y|X}(y|x)f_X(x) \;=\; xe^{-xy}e^{-x} \;=\; xe^{-x(y+1)}\,.$$

Then, we get

$$f_Y(y) = \int\limits_0^\infty xe^{-x(y+1)}dx \;=\; \frac{xe^{-x(y+1)}}{-(y+1)}|_0^\infty + \int\limits_0^\infty \frac{e^{-x(y+1)}}{y+1}dx \;=\; \frac{1}{(y+1)^2}\,.$$

And hence, we get

$$f_{X|Y}(x|y) \;=\; \frac{f_{X,Y}(x,y)}{f_Y(y)} \;=\; (y+1)^2xe^{-x(y+1)}\,.$$

In particular for $y = 1$, this yields

$$\begin{aligned} P(X < 1|Y = 1) &= \int\limits_0^1 4xe^{-2x}dx \;=\; 4\left(\frac{xe^{-2x}}{-2}|_0^1 + \int\limits_0^1 \frac{e^{-2x}}{2}dx\right) \\ &= 4(-\frac{1}{2}e^{-2} - \frac{e^{-2}}{4} + \frac{1}{4}) \;=\; 1 - 3e^{-2}\,. \end{aligned}$$

4. Given is an urn with m balls, a of them are green and $m-a$ of them are red. With probability a/m we replace a red ball by a green ball. With probability $(m-a)/m$ we don't do anything. We repeat this process (such that the new replacements are independent of the previous replacements) and denote N_a the minimal number of steps we need to turn all balls green. Find the expected value of N_a in particular for $a = 1$.

Solution 1: We really count all the steps. Let X be the random variable such that $X = 1$ if we replace red by green and $X = 0$ if we replace green by red.

$$E[N_a] \; = \; P(X=1)E(N_a|X=1) + P(X=0)E(N_a|X=0) \; .$$

In both cases we start a new game. If $X = 1$, we have $a + 1$ green balls. If $X = 0$, we are as good of as before. In both cases we need an extra step to conclude. Thus, we get

$$E[N_a] \; = \; \frac{a}{m}(1+E(N_{a+1})) + \frac{m-a}{m}(1+E(N_a)) \; = \; 1 + \frac{a}{m}E(N_{a+1}) + \frac{m-a}{m}E(N_a) \; .$$

This yields

$$\frac{a}{m}E[N_a] \; = \; 1 + \frac{a}{m}E[N_{a+1}]$$

and hence

$$E[N_a] \; = \; \frac{m}{a} + E[N_{a+1}] \; .$$

For the case $a = m - 1$, we get

$$E[N_{m-1}] \; = \; \frac{m}{m-1} \; .$$

Thus for $a = 1$, we find

$$E[N_1] = \frac{m}{1} + E[N_2] \; = \; \frac{m}{1} + \frac{m}{2} + E[N_3] \; = \; \sum_{j=1}^{m} \frac{m}{j}$$

$$\cong m \int_1^m \frac{dx}{x} \; = \; m \ln m \; .$$

Actually there is a constant $\gamma > 0$ such that

$$\lim_{m \to \infty} \frac{\sum_{k=1}^{m} \frac{1}{k}}{\ln m} \; = \; \gamma$$

So we expect $\gamma m \ln m$ many steps to replace all green. For $a = 2$ and $m = 4$, we get

$$E[N_2] \; = \; \frac{4}{2} + E[N_3] \; = \; \frac{4}{2} + \frac{4}{3} \; = \; \frac{10}{3} \; .$$

Solution 2: We only count the steps M_a where we don't do anything. We use X as above. Then

$$E[M_a] \;=\; P(X=1)E(M_a|X=1) + P(X=0)E(M_a|X=0)\;.$$

However now, we have $E(M_a|X=1) = E[M_{a+1}]$ because we got a green ball and this action is not on our ticker. This yields

$$E[M_a] \;=\; \frac{a}{m}E[M_{a+1}] + \frac{m-a}{m}(1+E[M_a])\;.$$

Now, we get

$$E[M_a] \;=\; \frac{m-a}{a} + E[M_{a+1}]\;.$$

For $a = m-1$, we find $E(M_a|X=1) = E[M_m] = 0$ and thus $E[M_{m=1}] = \frac{1}{m-1}$. In this case, we get

$$\begin{aligned} E[M_a] = \frac{m-a}{a} + E_[M_{a+1}] \;&=\; \frac{m-a}{a} + \frac{m-(a+1)}{a+1} + \cdots + \frac{1}{m-1} \\ &= \sum_{k=a}^{m-1}\frac{m}{a} - (m-a)\;. \end{aligned}$$

Indeed,

$$E[M_a] + (m-a) \;=\; E[N_a]$$

because we have to turn the $(m-a)$ balls green. In that case we get

$$E[M_1] \;=\; m(\sum_{k=1}^{m-1}\frac{1}{k}) - m + 1 \cong \gamma m \ln m - m\;.$$

For $m = 8$ and $a = 6$, we get here

$$E[M_6] \;=\; \frac{8-6}{6} + \frac{8-7}{7} \;=\; \frac{20}{42} \;=\; \frac{10}{21}\;.$$

The following problems (in connection with the homework) should provide a good preparation for the exam.

1. p231 no 31 **Solution:** a) Let us calculate

$$E|X-a| = \int_0^A |x-a|\frac{dx}{a} \;=\; \int_0^a a-x\frac{dx}{A} + \int_a^A x-a\frac{dx}{A}$$
$$= \frac{a^2}{A} - \frac{a^2}{2A} + \frac{A^2}{2A} - \frac{a^2}{2A} - \frac{(A-a)a}{A}$$
$$= \frac{A}{2} - a + \frac{a^2}{A}\,.$$

Differentiation yields $0 = -1 + \frac{2a}{A}$, thus $a = \frac{A}{2}$. Hence for $\frac{A}{2}$ we have the minimal value (compare to $a = 0$ and $a = A$ where we have $A/2$ which is bigger than $A/4$.

b) We have

$$\frac{1}{\lambda}E|X-a| = \int_0^\infty |x-a|e^{-\lambda x}dx \;=\; \int_0^a (a-x)e^{-\lambda x}dx + \int_a^\infty (x-a)e^{-\lambda x}dx$$
$$= a(1-e^{-\lambda a}) - [\frac{xe^{-\lambda x}}{-\lambda} + \frac{e^{-\lambda x}}{-\lambda^2}]_0^a + [\frac{xe^{-\lambda x}}{-\lambda} + \frac{e^{-\lambda x}}{-\lambda^2}]_a^\infty - a[\frac{e^{-\lambda x}}{-\lambda}]_a^\infty$$
$$= a(1-e^{-\lambda a}) + 2\frac{\lambda a e^{-\lambda a} + e^{-\lambda a}}{\lambda^2} - \frac{1}{\lambda^2} - a\frac{e^{-\lambda a}}{\lambda}$$
$$= \frac{1}{\lambda^2} + \frac{2}{\lambda^2}e^{-\lambda a} + a(\frac{1}{\lambda} - 1)e^{-\lambda a}$$

Thus differentiation of $[2 + a(\lambda - \lambda^2)]e^{-\lambda a}$ in a yields

$$0 = (\lambda-\lambda^2)e^{-\lambda a} - a[2 + a(\lambda-\lambda^2)]e^{-\lambda a} \;=\; -e^{-\lambda a}(\lambda-\lambda^2)(a^2 + \frac{2a}{\lambda-\lambda^2} - 1)$$
$$= -e^{-\lambda a}(\lambda-\lambda^2)[(a + \frac{1}{\lambda-\lambda^2})^2 + \frac{1}{\lambda^2(1-\lambda)^2} - 1]$$

Hence for $\lambda = 1$ we find that $a = 0$ is optimal. Only for $\lambda \geq \frac{1+\sqrt{5}}{2}$, we find interesting solutions

$$a \;=\; \frac{1 \pm \sqrt{\lambda^2(\lambda-1)^2 - 1}}{\lambda(\lambda-1)}\,.$$

What an ugly calculation. I don't know which one is minimal. By I suspect for large λ only the one with + makes sense.

2. p291 15 **Solution:** a) Well

$$\int_R cd(xy) = cArea(R) .$$

Hence $c = Area(R)^{-1}$. Now, we assume for b) that R is the rectangle $[-1,1]\times[-1,1]$. Then, we find

$$f_X(x) = \int_{-1}^{1} \frac{dy}{4} = \frac{1}{2}$$

whenever $-1 \leq x \leq 1$. Moreover, $f_Y(y) = \frac{1}{2}$ for $-1 \leq y \leq 1$. Therefore

$$f_X(x) f_Y(y) = \frac{1}{4} f_{X,Y}(x,y)$$

whenever $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ and 0 else. Thus X and Y are independent. (Only for a square though not for a circle). For problem b), we note that the circle is contained in the square and hence

$$P(X^2 + Y^2 \leq 1) = \frac{\pi}{4} .$$

3. p295 54

Solution: a) $U = XY$, $V = X/Y$

$$P(U \leq u, V \leq v) = \int_{1\leq x, 1\leq y, xy\leq u, x/y\leq v} \frac{dxdy}{x^2y^2} = \int_1^{\infty} \int_{1\leq x\leq \min(\frac{u}{y}, yv)} \frac{dx}{x^2}\frac{dy}{y^2}$$

$$= \int_1^{\infty} \int_{1\leq x\leq \min(\frac{u}{y}, yv)} \frac{dx}{x^2}\frac{dy}{y^2} = \int_1^{u} [1 - \min(\frac{u}{y}, yv)^{-1}]\frac{dy}{y^2}$$

$$= \int_{\max(1,\frac{1}{v})}^{u} [1 - \max(\frac{y}{u}, \frac{1}{yv})]\frac{dy}{y^2} .$$

Here we assume $u > 1$, if not we get $P(U \leq u, V \leq v) = 0$. Now, we have to discuss some cases. First, we note that $\max(y/u, 1/(yv)) = y/u$ if and only if $y^2 \geq u/v$. Let us first assume that $1/v > u$. Then there are no y's to choose from and hence $P(U \leq u, V \leq v) = 0$. So, we have to consider $1/v \leq u$ and get for $m = \max(1, 1/v)$:

$$\begin{aligned}
\int_{\max(1,\frac{1}{v})}^{u} [1 - \max(\frac{y}{u}, \frac{1}{yv})]\frac{dy}{y^2} &= \int_{\max(1,\frac{1}{v})}^{\sqrt{\frac{u}{v}}} [1 - \frac{1}{yv}]\frac{dy}{y^2} + \int_{\sqrt{\frac{u}{v}}}^{u} [1 - \frac{y}{u}]\frac{dy}{y^2} \\
&= [\frac{1}{2vy^2} - y^{-1}|_{y=\max(1,\frac{1}{v})}^{\sqrt{\frac{u}{v}}} + [\frac{\ln y}{u} - y^{-1}]_{y=\sqrt{\frac{u}{v}}}^{y=u} \\
&= \frac{1}{2u} - \sqrt{\frac{v}{u}} - \frac{1}{2vm^2} + \frac{1}{m} + \frac{\ln u}{u} - u^{-1} - \frac{\ln u + \ln v}{2u} + \sqrt{\frac{v}{u}} \\
&= \frac{1}{m} - \frac{1}{2vm^2} + \frac{\ln u - \ln v - 1}{2u}
\end{aligned}$$

However, this only holds of $u/v > 1$, i.e. $u > v$. Let us first consider $v < 1$. Then, we find $m = 1/v$ and hence

$$P(U \leq u, V \leq v) = -\frac{v}{2} + \frac{\ln u - \ln v - 1}{2u} .$$

Differentiation yields

$$f_{U,V}(u, v) = \frac{1}{2u^2 v} .$$

Now, if $1 < v < u$. Then $m = 1$ and

$$P(U \leq u, V \leq v) = 1 - \frac{1}{2v} + \frac{\ln u - \ln v - 1}{2u} .$$

Then differentiation yields again

$$f_{U,V}(u, v) = \frac{1}{2u^2 v} .$$

Finally if $v > u$. Then

$$P(U \leq u, V \leq v) = \int_{1}^{u} [1 - \frac{y}{u}]\frac{dy}{y^2} = \frac{\ln u}{u} - u^{-1} + 1$$

Then differentiation yields 0. Thus we get

$$f_{U,V}(u,v) \;=\; \begin{cases} \frac{1}{2vu^2} & \text{if } 1 \le u \text{ and } v \le 1 \text{ and } u \le \frac{1}{v} \text{ or } v \ge 1 \text{ and } u \ge v \\ 0 & \text{else} \end{cases} .$$

As for the marginal densities we have

$$f_U(u) \;=\; \int_{\frac{1}{v}}^{v} \frac{1}{2vu^2} dv \;=\; \frac{\ln u}{u^2}$$

for $u \ge 1$. Moreover, if $v > 1$, then

$$f_V(v) \;=\; \int_{v}^{\infty} \frac{1}{2vu^2} du \;=\; \frac{1}{2v^2}$$

and for $v < 1$

$$f_V(v) = \int_{1}^{\frac{1}{v}} \frac{1}{2vu^2} du = \frac{1-v}{2v} .$$

4. p387 62 That, we will discuss in class later.

5. Let X and Y be positive continuous random variables.

   (a) Show that

   $$f_{X/Y,Y}(x,y) \;=\; y f_{X,Y}(xy,y) .$$

   Solution: We deduce from $Y > 0$ that

   $$P(X/X \le a, Y \le b) = \int_{x/y \le a, y \le b} f_{X,Y}(x,y)dxdy \;=\; \int_0^b \int_0^{ya} f_{X,Y}(x,y)dxdy .$$

   Differentiation with respect to a and b and the chain rule yields.

   $$f_{X/Y,Y}(a,b) \;=\; b f X, Y(ab,b) .$$

(b) Assume in addition that X, Y are independent. Show that

$$f_{X/Y|Y}(x|y) \;=\; y f_X(yx)\,.$$

Solution: That's clear now, because by independence

$$f_{X/Y|Y}(x|y) = \frac{f_{X/Y,Y}(x,y)}{f_Y(y)} \;=\; \frac{y f_{X,Y}(xy,y)}{f_Y(y)} \;=\; \frac{y f_X(xy) f_Y(y)}{f_Y(y)} \;=\; y f_X(yx)\,.$$

(c) Let X, Y be a random variables such that the marginal density of Y is given by

$$f_Y(y) \;=\; \begin{cases} \frac{1}{y \ln 2} & \text{if } 1 \le y \le 2 \\ 0 & \text{else} \end{cases}\,.$$

X is exponential distributed with parameter Y. Determine $f_{X,Y}$ and f_X.

Solution: By assumption

$$f_{X|Y}(x|y) \;=\; y e^{-xy}$$

for $x \ge 0$ and hence

$$f_{X,Y}(x,y) \;=\; \begin{cases} \frac{e^{-xy}}{\ln 2} & \text{if } 1 \le y \le 2 \\ 0 & \text{else} \end{cases}\,.$$

Now, we find

$$\ln 2 f_X(x) = \int_1^2 e^{-xy} dy \;=\; [\frac{e^{-xy}}{-x}]_{y=1}^{y=2}$$
$$= \frac{e^{-x} - e^{-2x}}{x}\,.$$

(d) Let X and Y be independent such that X, Y has the exponential distribution with parameter μ, λ, respectively. Find $f_{X/Y}$.

Solution: According to a) and b), we know that

$$f_{X/Y,Y}(x,y) \;=\; y f_{X,Y}(xy,y) \;=\; y f_X(xy) f_y(y) \;=\; \lambda \mu y e^{-\mu x y} y e^{-\lambda y}\,.$$

This yields the marginal distribution

$$
\begin{aligned}
f_{X/Y}(x) = \int_{-\infty}^{\infty} f_{X/Y,Y}(x,y)dy \;&=\; \lambda\mu \int_{0}^{\infty} ye^{-\mu xy} ye^{-\lambda y} dy \\
= \lambda\mu \int_{0}^{\infty} ye^{-y(\lambda+x\mu)}dy \;&=\; \lambda\mu \left(\frac{ye^{-y(\lambda+x\mu)}}{-(\lambda+x\mu)}\Big|_{y=0}^{y=\infty} + \int_{0}^{\infty} \frac{e^{-y(\lambda+x\mu)}}{(\lambda+x\mu)}dy \right) \\
= \lambda\mu \frac{1}{(\lambda+x\mu)^2} \; .
\end{aligned}
$$

I don't know the name of this particular distribution, but it s not the Cauchy distribution.

6. Find the expectation of X where Z is $N(0,1)$ and $X(\omega) = Z(\omega)$ of $Z(\omega) > x_0$ and 0 else. Here we assume $x_0 > 0$.

   Solution: We see that $P(X \leq a) = 0$ if $a < 0$. Moreover, if $0 \leq a \leq x_0$, then $P(X \leq a) = P(X \leq x_0)$. Finally, for $a > x_0$

   $$P(X \leq a) \;=\; P(x_0 < Z \leq a) \; .$$

   We differentiate and get $f_X(a) = 0$ for all $a \neq 0$ and $f_X(a) = f_Z(a)$ for $a \geq x_0$. Integration yields

   $$\int_{-\infty}^{\infty} f_X(x)dx = Prob(Z > x_0)$$

   But that is not one! Indeed, X is a mixture of a continuous and a discrete random variable and $f_X(0) = P(Z \leq x_0)$. Therefore, we get (with $u = x^2/2$, $du = xdx$)

   $$
   \begin{aligned}
   E[X] = 0P(Z \leq x_0) + \int_{x_0}^{\infty} xf_Z(x)dx \\
   = \frac{1}{\sqrt{2\pi}} \int_{x_0}^{\infty} xe^{-\frac{x^2}{2}}dx \;&=\; \frac{1}{\sqrt{2\pi}} \int_{x_0^2/2}^{\infty} e^{-u}du \\
   = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_0^2}{2}} \; .
   \end{aligned}
   $$