Exam2-Math 444-April 14, 2008

There are 100 points in this exam.

Good luck

(1) (10P) State the mean value theorem.

Let $f : [a,b] \to \mathbb{R}$ be continuous and diff';e on (a,b). Then there exists $a < x < b$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(x) .$$

(2) (10) Which of the following results is true for a sequence (f_n) of differentiable functions on an open Interval I

(a) If (f_n) converge uniformly to f and (f_n) converge pointwise to g, then $f'(x) = g(x)$ for all $x \in I$.

(b) If (f_n) converge pointwise to f and (f_n) converge uniformly to g, then $f'(x) = g(x)$ for all $x \in I$.

The first one is wrong (that is hard to show, but $f_n(x) = \min(|1-x|^n, |1+x|^n)$ on $(-1,1)$ does the trick because the limiting function is not continuous. The second one is correct.

(3) (15P) Show that $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$ and calculate the derivative.

The function $g(x) = x^2$ is differentiable and has the derivative $g'(x) = 2x$. Since $g'(x) > 0$ for $x > 0$, we know that g is strictly increasing. Thus the theorem about inverse functions applies and yields

$$(g^{-1})'(y) = \frac{1}{g'(x)}$$

whenever $y = g(x)$. Thus for $y = g(x) = x^2$ we get

$$(g^{-1})'(y) = \frac{1}{2x} = \frac{1}{2\sqrt{y}}\,.$$

Thus the derivative of $f(y) = \sqrt{y}$ is given by $\frac{1}{2\sqrt{y}}$.

(4) (30P) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 1$ and

$$f'(x) \;=\; xf(x)$$

holds for all $x \in \mathbb{R}$.

a) Show that $f(x) > 0$ for all $x > 0$. (Hint: Proceed by contradiction and consider

$$x_0 \;=\; \inf\{x > 0 : f(x) \;\le\; 0\}\,.$$

The mean value theorem will allow you to obtain a contradiction.)

b) Show that f is strictly monotone on $(0, \infty)$.

Solution with the x_0 above, we get

$$\frac{f(x_0) - f(1)}{x_0} \;=\; f'(z) \;=\; f(z)z$$

for some z between 0 and x_0. This means

$$f(x_0) \;=\; f(1) + x_0 f(z)z > 0$$

because $f(z) > 0$. It follows from the intermediate value theorem that $f(x_0) = 0$. This contradiction shows that there is no $x > 0$ such that $f(x) \le 0$.

For the part b) we deduce the assertion from the fact that $f'(x) > 0$ for all $x > 0$.

(5) (30P) Let K and B be compacts sets in $\mathbb{R}$ such that

$$K + B = \{x + y : x \in K, y \in B\}$$

is compact.

Let $z_n = x_n + y_n \in K + B$. Let (x_{n_k}) be a convergent subsequence and $y_{n_{k_j}}$ be a convergent subsequence. Then

$$z = \lim_j x_{n_{k_j}} + y_{n_{k_j}} = x + y$$

is also in $K + B$.