1. Study problems

(1) Let $K \subset \mathbb{R}$ be a compact set and $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $f(K)$ is compact. Given an example of a set $T \subset \mathbb{R}$ and f such that $f(T)$ is compact, but T is not. Is it possible to find such a counterexample with f strictly increasing?

(2) Show that $\{x : \frac{1}{3} \leq \sin(x) \leq \frac{2}{3}\}$ is closed.

(3) Close the book and prove the intermediate value theorem. Hint consider the set $S = \{x | a \leq x \forall a \leq y \leq x : f(y) \leq m\}$, where m is the intermediate value you want to reach.

(4) Assuming Rolle's Lemma, proof the mean value theorem.

(5) Let (f_n) be a sequence of continuous functions converging uniformly to g. Show that g is continuous. Reformulate our theorem about differentiation and limits with the help of this information.

(6) Let $f_n(x) = n \sin(\frac{x}{n})$. Is the family (f_n) uniformly continuous?

(7) Let $f : [0, 10] \to \mathbb{R}$ be a continuous function, differentiable on (a, b) such that $f(0) = 0$ and $f'(x) \leq 3$. Show that $f(x) \leq 30$.

(8) Show that $f(x) = \frac{\sin(x)}{\cos(2x)}$ is monotone on $[0, \frac{\pi}{8}]$ and calculate the derivative of the inverse function. (You may use some facts about sin and cos we haven't proved in class).

(9) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions and differentiable on (a, b). Show that there exists an x such that

$$[f(b) - f(a)]g'(x) \;=\; [g(b) - g(a)]f'(x)\,.$$

Do you have to assume $g'(x) \neq 0$?

(10) Show that

$$f(x) \;=\; \sum_k \frac{1}{k^3} \cos(kx)$$

is differentiable.