Homework 468

Due: 8/30

Let X be a vector space. Subset $K \subset X$ is called convex if for all $x, y \in K$ and $0 \leq \lambda \leq 1$

$$(1-\lambda)x + \lambda y \in \mathcal{K}$$

Let $K \subset X$ be convex, then a function $\phi : K \to \mathbb{R}$ is convex if for all $x, y \in K$ and $0 \leq \lambda \leq 1$

$$\phi((1-\lambda)x + \lambda y) \ \leq \ (1-\lambda)\phi(x) + \lambda\phi(y) \ .$$

1. Let $1 \leq p < \infty$ and consider the following expression

$$\|(x_k)\|_{v_p} \ = \ \sup_{(k_j) \text{ subseqeunce}} \left(|x_{k_1}|^p + \sum_{j=1}^{\infty} |x_{k_{j+1}} - x_{k_j}|^p \right)^{\frac{1}{p}} \ .$$

   Let V_p be the space of sequences such that $\|(x_k)\|_{v_p}$ is finite. Show that $(V_p, \| \, \|_{v_p})$ is a Banach space.

2. Let $K \subset X$ and $L \subset Y$ be convex sets and $\Phi : K \times L \to \mathbb{R}$ be convex. Show that $\phi : K \to [\infty, \infty)$ defined by

$$\phi(x) \ = \ \inf_{y \in L} \Phi(x, y)$$

   is convex.

3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex with $\phi(0) = 0$. Show that for $s < t$

$$\frac{\phi(s)}{s} \ \leq \ \frac{\phi(t)}{t} \ .$$

   Show that

$$\phi'_+(0) \ = \ \lim_{t>0, t \to 0} \frac{\phi(t)}{t}$$

   exists and

$$\phi(t) \ \geq \ t\phi'_+(0) \ .$$

4. Let X be a vector space $f : X \to \mathbb{R}$ be convex, $V \subset X$ be a subspace and $l : V \to \mathbb{R}$ be a linear map such that $l \leq f$ on V. Let $x \in X \setminus V$ and V_1 be the subspace of X spanned by V and x. Show that there exists an extension l_1 of l to V_1 such that still $l_1 \leq f$ on V_1. (Hint consider the map $\Phi : \mathbb{R} \times V \to \mathbb{R}$ defined by $\Phi(t, y) \ = \ f(tx + y) - l(y)$.)

Math 468-Banach spaces, No 2.

Due: 9/6/2000

1. Show that
$$d(\ell_1^n, \ell_2^n) \;\le\; \sqrt{n}\,.$$

2. Show that for two n-dimensional spaces E and F
$$d(E,F) \;=\; d(E^*,F^*)\,.$$

3. Let $1 \le p, q \le \infty$ and α be the norm on matrices defined by
$$\alpha(a) \;=\; \left(\sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{p}} \,.$$
Calculate α^*.

4. Let α be the norm of $n \times n$ matrices given by
$$\alpha(a) \;=\; \sum_{j=1}^k b_j$$
where $1 \le k \le n^2$ and $b_1, ..., b_{n^2}$ provides the list of elements $\{|a_{11}|, ..., |a_{nn}|\}$ such that $b_1 \ge b_1 \ge ... \ge b_{n^2}$. Calculate α^*.

5. Prove the arithmetic/ geometric mean inequality
$$(\prod_{j=1}^n a_j)^{\frac{1}{n}} \;\le\; \frac{1}{n} \sum_{j=1}^n a_j$$
and discuss equality. (Hint: $f(x) = -\log(x)$ is convex.)

Math 468-Banach spaces, No 3.

Due: 9/13/2000

Notation: Let $(x_j) \subset X$ be a sequence in a Banach space X. We put
$$\|(x_j)\|_p \;=\; \left(\sum_{j=1}^\infty \|x_j\|_X^p \right)^{\frac{1}{p}} \,.$$

and

$$w_p((x_j)) \;=\; \sup_{\|x^*\|_{X^*}\le 1} \left(\sum_{j=1}^{\infty} |x^*(x_j)|^p \right)^{\frac{1}{p}} .$$

Let $\frac{1}{p} + \frac{1}{p'} + 1$. For a linear map $T : X \to Y$, we put

$$\nu_p(T) \;=\; \inf \left\|(x_j^*)\right\|_{p'} w_p((y_j)) ,$$

where the infimum is taken over all sequences $(x_j^*) \subset X^*$, $(y_j) \subset X$ such that

$$T(x) \;=\; \sum_{j=1}^{\infty} x_j^*(x) y_j . \qquad (1)$$

We denote by $N_p(X,Y)$ the set of linear maps $T : X \to Y$ such that $\nu_p(T) < \infty$.

1. Show that $(N_p(X,Y), \nu_p)$ is a Banach space. (Hint: Use $st = \inf_{r>0} \frac{1}{p}(rs)^p + \frac{1}{p'}(rt^{-1})^{p'}$ and the fact that for $T \in N_p(X,Y)$, $\varepsilon > 0$, one can choose (x_j^*), (y_j) satisfying (1) such that

$$\left\|(x_j^*)\right\|_{p'}^{p'} \;=\; w_p((x_j))^p \;\le\; (1+\varepsilon)\nu_p(T) .$$

2. (a) Let $x_1, ..., x_n \in \ell_2$. Show that

$$\frac{1}{2^n} \sum_{\varepsilon_1=\pm 1,...,\varepsilon_n=\pm 1} \left\| \sum_{k=1}^{n} \varepsilon_k x_k \right\|_2^2 \;=\; \sum_{k=1}^{n} \|x_k\|_2^2 .$$

   (b) Let X be a Banach space and $T : \ell_\infty^n \to X$ be a linear map. Show that

$$\sum_{j=1}^{n} \|T(x_j)\|_X \;=\; \sup_{\|S:X\to\ell_\infty^n\|\le 1} |tr(TS)| .$$

   (c) Let $T : \ell_\infty^n \to \ell_2^n$. Show that

$$\sum_{j=1}^{n} \|T(e_j)\|_2 \;\le\; \sqrt{n}\,\|T\| .$$

   (d) Show

$$d(\ell_2^n, \ell_\infty^n) \;\ge\; \sqrt{n}.$$

Math 468-Banach spaces, No 4.

Due: 9/20/2000

1. Let $T : X \to X$ be a finite rank map and let $(x_j^*)_{j=1}^m$, $(x_j)_{j=1}^m$, $(y_j^*)_{j=1}^n$, $(y_j)_{j=1}^n$, such that for all $x \in X$

$$T(x) \;=\; \sum_{j=1}^m x_j^*(x) x_j \quad \text{and} \quad T(x) \;=\; \sum_{j=1}^n y_j^*(x) y_j \;.$$

Show

$$\sum_{j=1}^m x_j^*(x_j) \;=\; \sum_{j=1}^n y_j^*(y_j) \;.$$

2. Let $\alpha = \sum_j \alpha_j$ be summable sequence of real numbers and $\varepsilon > 0$. Find a sequence (η_j) such that

$$\lim_j \eta_j \;=\; 0 \quad \text{and} \quad \sum_j \eta_j^{-1} \alpha_j \;\leq\; (1+\varepsilon)\alpha \;.$$

3. On $\mathbb{K}^n$ we define the norm

$$\|x\| \;=\; \left(a_1^2 + a_2^2\right)^{\frac{1}{2}} \;,$$

$a_1, ..., a_n$ is the non-decreasing arrangement of $\{|x_1|, ..., |x_n|\}$, i.e. $a_1 \geq a_2 \geq \cdots \geq a_n$ and all the values $|x_1|, ..., |x_n|$ appear in that list. Show that $\| \; \|$ is a norm. Find the intersection of this unit ball with the $x = y$ plane. Calculate the norm of the functional $\phi : (\mathbb{R}^3, \| \; \|) \to \mathbb{R}$, $\phi(x) = x_1 + x_2 + x_3$.

Math 468-Banach spaces, No 5.

Due: 9/27/2000

1. Let $(x_n) \subset X$ be a sequence in a Banach space such that
$$\lim_n \|x_n\| \; = \; 0 \,.$$
Show that the absolute convex hull
$$\{\sum_n \alpha_n x_n \; : \; \sum_n |\alpha_n| \; \leq \; 1\}$$
is compact.

2. Let $C \subset X$ be a compact set
   (a) Show that $2C$ has a finite ε-net.
   (b) Let $\{x_1, ..., x_N\}$ be an ε-net for $2C$. Show that
   $$C_2 \; = \; \bigcup_{i=1}^{N} [(B(x_i, \varepsilon) \cap 2C) - x_i]$$
   is again compact.
   (c) Find inductively finite 4^{-n} nets Δ_{n+1} for
   $$C_{n+1} \; = \; \bigcup_{y \in \Delta_n} [(B(y, 4^{-n}) \cap 2C_n) - y] \,.$$
   Show that for $x \in C$ there are $(y_j)_{j=1}^n$ such that $y_j \in \Delta_j$ and
   $$x - (\sum_{j=1}^{n} 2^j y_j) \in C_{n+1} \,.$$
   Show $\|y_j\| \; \leq \; 24^{-j+1}$ for all j.
   (d) Conclude that every compact set is in the closed convex hull of a sequence converging to 0.

3. (a) Let $a, b \geq 0$ and show that
$$\sup_{x^2+y^2 \leq 1, x \geq y \geq 0} ax + by \; = \; \begin{cases} \sqrt{a^2+b^2} & \text{if} a \geq b \\ \frac{a+b}{\sqrt{2}} & \text{if} a \leq b \end{cases} \,.$$

(b) For a sequence $(x_1, .., x_n)$ of scalars,we denote by $x_1^* \geq x_1^* \geq ... \geq x_n^*$ the non-decreasing rearrangement (of the absolute values). Use the inequality

$$|\sum_{i=1}^{n} x_i y_i| \leq \sum_{i=1}^{n} x_i^* y_i^*$$

to show that for $y_1, ..., y_n \in \mathbb{K}$

$$\|y\|_* = \sup_{(x_1^*)^2+(x_2^*)^2 \leq 1} \left| \sum_i x_i y_i \right| = \begin{cases} \sqrt{(y_1^*)^2 + (\sum_{i=2}^{n} y_i^*)^2} & \text{if } y_1^* \geq \sum_{i=2}^{n} y_i^* \\ \frac{\sum_{i=1}^{n} y_i^*}{\sqrt{2}} & \text{if } y_1^* \leq \sum_{i=2}^{n} y_i^* \end{cases} .$$

If not done so calculate the norm of $\|(1, 1, 1)\|_*$.

Math 468-Banach spaces, No 6.

Due: 9/4/2000

1. Show that

$$|\sum_{i=1}^{n} x_i y_i| \;\leq\; \sum_{i=1}^{n} x_i^* y_i^*$$

Where $(x_1^*, ..., x_n^*)$, $(y_1^*, ..., y_n^*)$, is the non-decreasing rearrangement of $(|x_1|, ..., |x_n|)$, $(|y_1|, ..., |y_n|)$, respectively. Hints: There is no loss of generality to assume that $x_1 = x_1^*$,..., the same for the $y_1, ..y_n$ and the for all permutations we have to show

$$\sum_{i=1}^{n} x_i y_{\pi(i)} \;\leq\; \sum_{i=1}^{n} x_i y_i$$

Observe that for any π

$$\sum_{j=1}^{k} y_{\pi(j)} \leq \sum_{j=1}^{k} y_j$$

and use the Abel summation

$$\sum_{i=1}^{n} x_i y_{\pi(i)} \;=\; \sum_{i=1}^{n} x_i (\sum_{j=1}^{i} y_{\pi(j)} - \sum_{j=1}^{i-1} y_{\pi(j)}) \;=\; \cdots \;.$$

2. Let α be the norm of $n \times n$ matrices given by

$$\alpha_k(a) \;=\; \sum_{j=1}^{k} b_j$$

where $1 \leq k \leq n^2$ and $b_1, ..., b_{n^2}$ provides the list of elements $\{|a_{11}|, ..., |a_{nn}|\}$ such that $b_1 \geq b_1 \geq ... \geq b_{n^2}$. Calculate α_k^*.

Math 468-Banach spaces, No 7.

Due: 10/11/2000

1. Let (Ω, Σ, μ) be a measure space and $A_1, ..., A_m$ be disjoint sets in Σ such that $\mu(A_i) > 0$ for $i = 1, .., m$. We define the map

$$E(f)(\omega) \;=\; \sum_{i=1}^{m} \int_{A_i} f \frac{d\mu}{\mu(A_i)} 1_{A_i}(\omega)\ .$$

Recall

$$1_A(\omega) \;=\; \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{else} \end{cases}\ .$$

Show for $1 \le p \le \infty$ and $f \in L_p(\Omega, \Sigma, \mu)$

$$\|E(f)\|_p \;\le\; \|f\|_p\ .$$

Let

$$X_p \;=\; \left\{ \sum_{i=1}^{m} \alpha_i 1_{A_i} \;:\; \alpha_i \in \mathbb{K} \right\} \subset L_p(\Omega, \Sigma, \mu)\ .$$

Show that

$$\|f - E(f)\|_p \;=\; \inf\{\|f - g\|_p \;:\; g \in X_p\}\ .$$

2. Let X, Y Banach spaces. We say that a net (T_i) converges to T in the topology τ if for every compact set $K \subset X$

$$\lim_i \sup_{x \in K} \|T_i(x) - T(x)\|_Y \;=\; 0\ .$$

Let $(y_n)^* \subset Y^*$ and $(x_n) \subset X$ such that

$$\sum_n \|y_n^*\|_{Y^*} \|x_n\|_X \;<\; \infty\ .$$

Show that

$$\phi(T) \;=\; \sum_n y_n^*(T(x_n))$$

is continuous with respect to τ. (Hint you can assume that $\lim_n \|x_n\| = 0$, why?). Let $K \subset X$ be a compact set and (x_n) a sequence tending to 0 such that $K = \overline{conv}\{x_n\}$ (Why can we assume that?). Consider

$$\Psi : L(X, Y) \to c_0(Y)$$

defined by

$$\Psi(T) \;=\; (T(x_n))\;.$$

Show that

$$\|\Psi(T)\| \;=\; \sup_{x\in K} \|Tx\|\;.$$

Use the Hahn-Banach theorem to show that for every functional $\phi : \mathcal{L}(X,Y) \to \mathbb{K}$ with

$$|\phi(T)| \;\le\; \sup_{x\in K} \|Tx\|$$

there are $(y_n^*) \subset Y^*$ such that for all T

$$\phi(T) \;=\; \sum_n y_n^*(T(x_n))$$

and

$$\sum_n \|y_n^*\| \;<\; \infty\;.$$

(Hint, $c_0(Y)^* = \ell_1(Y^*)$.)

3. Let X, Y be Banach spaces. Let $X^* \subset C(B_{X^{**}}), Y \subset C(B_{Y^*})$ be the canonical embeddings. For a finite rank map $S = \sum\limits_{i=1}^{n} x_i^* \otimes y_i$, we define

$$\Psi(S)(x^{**}, y^*) \;=\; \sum_{i=1}^{n} x^{**}(x_i) y^*(y_i)\;.$$

Show that

$$\|\Psi(S)\|_{C(B_{X^{**}} \times B_{Y^*})} \;=\; \|S\|\;.$$

What does the Hahn-Banach tell you about an integral map $T : Y \to X$?

Math 468-Banach spaces, No 8.

Due: 10/18/2000

1. Let $1 \le p < q \le\le \infty$ and $T : X \to Y$ be an absolutely p-summing map. Show that
$$\pi_q(T) \ \le \ \pi_p(T) \ .$$
(Hint: Choose r such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and observe that for $x_1, ..., x_n \in X$
$$\left(\sum_{k=1}^n \|T(x_k)\|_Y^q \right)^{\frac{1}{q}} \ = \ \sup_{\sum_k |\tau_k|^r \le 1} \left(\sum_{k=1}^n \|T(\tau_k x_k)\|_Y^p \right)^{\frac{1}{p}} \ ,$$
where the supremum is taken over positive real numbers $\tau_1, .., \tau_n$.

2. Let $2 \le p \le \infty$ and $i_{2,p}^n : \ell_2^n \to \ell_p^n$ be the identity map. Show
$$\pi_2(i_{2p}^n) \ = \ \sqrt{n} \quad \text{and} \quad \pi_2((i_{2p}^n)^{-1}) \ = \ \sqrt{n} \ .$$
Find the ellipsoid of maximal volume in the unit ball of ℓ_p^n.

3. Let $\mathbb{K} = \mathbb{R}$ and $i_{1,2}^n : \ell_1^n \to \ell_2^n$ be the identity map. Show
$$\pi_2(i_{1,2}^n) \ = \ 1 \ .$$
(Hint: Consider $S = \{-1, 1\}^n$ and $\mu(\{\varepsilon_1, ..., \varepsilon_n\}) = 2^{-n}$ the normalized counting measure. Then the map $i(x)(\varepsilon) = \sum_{i=1}^n \varepsilon_i x_i$ is an isometric embedding of ℓ_1^n in $L_\infty(S, \mu)$.)

Math 468-Banach spaces, No 9.

Due: 10/25/2000

1. Our goal is to show in the real case
$$\pi_1(id_{\ell_1^n}) \; > \; \sqrt{n} \, .$$
   (a) Let ν be the Haar measure on $\{-1,1\}^n$, i.e.
   $$\nu(\{(\varepsilon_1, ..., \varepsilon_n)\}) \; = \; 2^n \, .$$
   Show that
   $$\int_{\{-1,1\}^n} |\sum_{i=1}^n \varepsilon_i| \, d\nu(\varepsilon) \; < \; \sqrt{n} \, .$$
   (b) Let $c = \pi_1(id_{\ell_1^n})$. Show that c is the smallest constant such that for all $x_1, ..., x_n \in \mathbb{R}$
   $$\sum_{i=1}^n |x_i| \; \le \; c \int_{\{-1,1\}^n} |\sum_{i=1}^n x_i \varepsilon_i| \, d\nu(\varepsilon) \, .$$
   (Hint: By averaging an arbitrary measure can be replaced by the Haar measure and ℓ_1^n is isomterically embedded in $\ell_\infty(\{-1,1\}^n)$).

   (c) Conclude $\pi_1(id_{\ell_1^n}) > \sqrt{n}$.

2. Show that in the real case $\pi_1(id_{\ell_2^n})$ is the smallest constant c such that for all $x_1, ..., x_n \in \mathbb{R}$
$$\left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}} \; \le \; c \int_{S^{n-1}} |\sum_{i=1}^n x_i y_i| \, d\sigma(y) \, ,$$
where σ is the rotation invariant normalized surface measure on the sphere S^{n-1}. Conclude again $\pi_1(id_{\ell_1^n}) > \sqrt{n}$.

3. Let (α_n) be a sequence in ℓ_2 of norm less than one. Show that there exists a Banach space X and vectors $(x_n) \subset X$ such that
$$\alpha_n \; = \; \|x_n\|_X$$
and
$$\sup_{\varepsilon_n = \pm 1} \left\| \sum_n \varepsilon_n x_n \right\|_X \; \le \; 1 \, .$$

Math 468-Banach spaces, No 10.

Due: 11/1/2000

1. Let $2 \leq p < \infty$. A subset $S \subset \mathbb{Z}$ is called Λ_p set if there exists a constant $c > 0$ such that for all finite sequences of scalars

$$\left(\int_0^{2\pi} |\sum_{k \in \Lambda} \alpha_k e^{ikt}|^p \frac{dt}{2\pi} \right)^{\frac{1}{p}} \leq c \left(\sum_{k \in \Lambda} |\alpha_k|^2 \right)^{\frac{1}{2}} .$$

Then $\Lambda_p(S) = \inf c$, the infimum is taken over all possible constant. Let $P_\Lambda : C([0, 2\pi]) \to \ell_2(S)$ be the map

$$P_S(f) = (\hat{f}(k))_{k \in S} ,$$

where

$$\hat{f}(k) = \int_0^{2\pi} f(t) e^{-ikt} \frac{dt}{2\pi} .$$

Let $1 \leq p' \leq 2$ be defined by $\frac{1}{p} + \frac{1}{p'} = 1$. Show that

$$\Lambda_p(S) = \Pi_{p'}(P_S) .$$

2. Show that for all finite sequences (α_k)

$$\left(2^{-n} \sum_{\varepsilon_k = \pm 1} |\sum_{k=1}^n \alpha_k \varepsilon_k|^4 \right)^{\frac{1}{4}} \leq 3 \left(\sum_{k=1}^n |\alpha_k|^2 \right)^{\frac{1}{2}} .$$

(I hope 3 is okay it might be 5, though)

Math 468-Banach spaces, No 10.

Due: 11/8/2000

1. Let $h_1, .., h_n$ be positive, measurable functions such that for all $i = 1, .., n$

$$\int_{\mathbb{R}} h_i(x)\, dx \;=\; 1$$

and

$$\int_{\mathbb{R}} |x| h_i(x)\, dx \;<\; \infty$$

and $h_i(x) = h_i(-x)$. Let $y_1, ..., y_n$ be vectors in a Banach space Y. Show

$$\int_{\mathbb{R}^n} \sup_{i=1,..,n} |x_i|\, \|y_i\|\, h_1(x_1) \cdots h_n(x_n)\, dx_1, ..., dx_n$$

$$\leq \int_{\mathbb{R}^n} \left\| \sum_{i=1}^{n} x_i y_i \right\|_Y h_1(x_1) \cdots h_n(x_n)\, dx_1, ..., dx_n\ .$$

2. Let $h_1, ..., h_n$ be as above and

$$\gamma_p(i) \;=\; \left(\int_{\mathbb{R}} |x|^p h_i(x)\, dx \right)^{\frac{1}{p}} < \infty$$

and

$$\gamma_p \;=\; \sup_{i=1,..,n} \gamma_p(i)\ .$$

Show

$$\int_{\mathbb{R}^n} \sup_{i=1,..,n} |x_i| h_1(x_1) \cdots h_n(x_n)\, dx_1, ..., dx_n \;\;\leq\;\; n^{\frac{1}{p}} \gamma_p\ .$$

Math 468-Banach spaces, No 12.

Due: 11/15/2000

1. $n, m \in \mathbb{N}$ Show that

$$\int \left\| \sum_{i=1,\dots n, j=1,\dots m} x_{ij} e_i \otimes e_j : \ell_2^n \to \ell_2^m \right\| \, d\gamma_{nm}(x) \;\le\; C\sqrt{n+m}\,.$$

   Hint: Using Δ-nets this follows from the deviation inequality considering the random variables

$$F_{y,z}(x) \;=\; |\sum_{ij} x_{ij} z_i y_j|\,.$$

2. Let $2 \le p < \infty$ Show

$$\int \sup_{k=1,..,n} x_k^* \, d\gamma(x) \sim_{c(p)} (\ln n)^{1/2}\,.$$