3. Connection to the Riemann integral

Definition 3.1. *Let f be bounded. The* oscillation *of f at x is defined by*

$$\omega(x) = \lim_{\delta \to 0} \sup\{|f(z) - f(y)| : |z - x| < \delta, |y - x| < \delta\}$$

Remark 3.2. *$\omega(x)$ well-defined. In particular, $\omega(x) \leq 2 \sup |f| < \infty$.*

Remark 3.3. *f continuous at $x \Leftrightarrow \omega(x) = 0$*

modcont Remark 3.4. $m(\{x : \omega(x) \neq 0\}) = \lim_{n\to\infty} m(\{x : \omega(x) > 1/n\})$

Definition 3.5. *(Darboux Integral)*
For a partition $\pi =< a = x_0, x_1, \ldots, x_n = b >$ we define the upper and the lower sum by

$$\overline{S}(\pi, f) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{[x_{i-1}, x_i]} f$$
$$\underline{S}(\pi, f) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f$$

A bounded function f is called Darboux-integrable if $\forall \epsilon > 0\ \exists$ partition π such that

$$\overline{S}(\pi, f) - \underline{S}(\pi, f) < \epsilon$$

Remark 3.6. *A function is Darboux-integrable if and only if it is Riemann-integrable (see Royden for a precise definition).*

Proposition 3.7. *A bounded function is Riemann integrable on $[a, b]$ if and only if the set of discontinuity points has measure 0.*

Proof. "$\Longrightarrow$": Let f be Darboux-integrable. Let $\gamma > 0$ and $\delta > 0$. We define $\epsilon = \gamma\delta$. By definition there exists a partition $\pi =< 0 = x_0, x_1, \ldots, x_n = 1 >$ such that

$$\overline{S}(\pi, f) - \underline{S}(\pi, f) < \epsilon.$$

Consider $x \in (x_i, x_{i+1})$ such that $\omega(x) \geq \gamma$. Then

$$\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f \geq \gamma.$$

Observe that points with 'large' (and hence certain discontinuity) points satisfy

$$\{x : \omega(x) \geq \gamma\} = \bigcup_{i, \sup f - \inf f \geq \gamma} (x_i, x_{i+1}) \cup \{x_0, \ldots, x_n\}$$

Hence,

$$
\begin{aligned}
m(\{x:\omega(x)\geq\gamma\}) &\leq \sum_{\sup_{[x_i,x_{i+1}]} f-\inf_{[x_i,x_{i+1}]} f\geq\gamma} |x_{i+1}-x_i| \\
&\leq \frac{1}{\gamma}\sum |x_{i+1}-x_i|(\sup_{[x_i,x_{i+1}]} f-\inf_{[x_i,x_{i+1}]} f) \\
&\leq \frac{1}{\gamma}\left(\overline{S}(\pi,f)-\underline{S}(\pi,f)\right) \\
&\leq \frac{\epsilon}{\gamma}=\delta
\end{aligned}
$$

Since δ is arbitrary, we get $m(\{x:\omega(x)\geq\gamma\})=0$. However, $\gamma>0$ is arbitrary and thus Remark (3.4) implies $m(\{x:\omega(x)>0\})=0$. This means the set of discontinuity points has measure 0.

"$\Longleftarrow$" Define $\omega_\delta(x)=\sup\{|f(z)-f(y)|:|z-x|<\delta,|y-x|<\delta\}$.
By assumption we have $m(\{x:\omega(x)>0\})=0$. This implies

$$m(\{x:\omega(x)\geq\gamma\})=0$$

for all $\gamma>0$. Therefore, we deduce from the monotonicity of $\omega_\delta(x)$ that

dd (3.1) $$m(\{x:\omega(x)\geq\gamma\})=m(\{x:\lim_{\delta\to0}\omega_\delta(x)\geq\gamma\}) \;=\; \lim_{\delta\to0} m(\{x:\omega_\delta(x)\geq\gamma\})=0$$

Let $\gamma>0$ and $\epsilon=\frac{\gamma}{2\sup|f|}$. By (3.1) we deduce the existence of some k such that

$$m(\{x:\omega_{1/k}(x)\geq\gamma\})<\epsilon\,.$$

Choose $m>k$ and $\pi=<x_0,x_1,\ldots,x_m>=<0,\frac{1}{m},\frac{2}{m},\ldots,\frac{m}{m}>$.
Define

$$S \;=\; \{j\in\{1,\ldots,m\}:\exists_{x\in[x_{j-1},x_j]}\;\;\omega_{1/k}(x)\geq\gamma\}\,.$$

Then we get

$$\sum_{j\in S}(\sup_{[x_i,x_{i+1}]} f-\inf_{[x_i,x_{i+1}]} f)(x_{j-1}-x_j)\leq\gamma\sum_{j\in S}|x_{j+1}-x_j| \;\leq\; \gamma(b-a)\,.$$

If $j\notin S$, then

$$[x_{j-1}-x_j]\subset\{x:\omega_{1/k}(x)>\gamma\}\,.$$

This implies

$$
\begin{aligned}
&\sum_{j\notin S}(\sup_{[x_i,x_{i+1}]} f-\inf_{[x_i,x_{i+1}]} f)(x_{j-1}-x_j)\leq 2\sup_{[0,1]}|f|\,m(\bigcup_{j\notin S}[x_{j-1},x_j]) \\
&\leq\; 2\sup_{[a,b]}|f|\,m(\{x:\omega_{1/k}(x)>\gamma\}) \;\leq\; 2\sup_{[a,b]}|f|\epsilon \;\leq\; \gamma\,.
\end{aligned}
$$

Putting the pieces together, we find

$$\sum_j \Big(\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f \Big)(x_{j-1} - x_j) \;\le\; \gamma(b-a) + \gamma \;=\; \gamma(1 + b - a)\,.$$

Since $\gamma > 0$ is arbitrary we deduce that f is Darboux integrable. ■