Math 361, Section E1, Fall 2002
Exam 3, December 6

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [35 points] Let X be a continuous random variable with density

$$f_X(t) = \begin{cases} 1 & \text{if } t \in (0,1] \\ 0 & \text{else} \end{cases}$$

(i.e., X is $U(0,1)$). Define the transformation

$$\varphi(u) \overset{\text{def}}{=} \begin{cases} 4 & \text{if } u \in (0, \frac{1}{10}] \\ -1 & \text{if } u \in (\frac{1}{10}, \frac{4}{10}] \\ 3 & \text{if } u \in (\frac{4}{10}, 1] \end{cases}$$

Define $Y \overset{\text{def}}{=} \varphi(X)$.

(a) [10 points] Compute F_Y, the cumulative distribution function of Y.

(b) [5 points] Is Y continuous, or is it discrete (Hint: in this case, it is one or the other).

(c) [10 points] Compute the density (i.e., the discrete density if Y is discrete, and the continuous density if Y is continuous).

(d) [10 points] Compute $\mathbb{E}[Y]$.

2. [31 points] Suppose that X and Y are independent continuous random variables with densities

$$f_X(s) \overset{\text{def}}{=} \begin{cases} \lambda e^{-\lambda s} & \text{if } s \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) \overset{\text{def}}{=} \begin{cases} \nu e^{-\nu t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

where λ and ν are positive parameters. Define $Z \overset{\text{def}}{=} \min\{X, Y\}$.

(a) [10 points] Compute $\mathbb{P}\{Y \geq X\}$.

(b) [10 points] Compute $\mathbb{P}\{Z > 5\}$.

(c) [3 points] Compute $\mathbb{P}\{Z \leq 5\}$.

(d) [8 points] Compute F_Z, the cumulative distribution function of Z.

3. [34 points] Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) \overset{\text{def}}{=} \begin{cases} 6t & \text{if } s \geq 0,\ t \geq 0, \text{ and } s + t \leq 1 \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} X + Y$.

(a) 10 points Compute f_Y, the second marginal of $f_{X,Y}$.

(b) 10 points Compute $\mathbb{P}\left\{Z \le \frac{1}{3}\right\}$.

(c) 7 points Compute F_Z, the cumulative distribution function of Z. Hint: F_Z should be continuous.

(d) 7 points Compute f_Z, the density of Z.

Answers

1. (a)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < -1 \\ \frac{3}{10} & \text{if } -1 \le t < 3 \\ \frac{9}{10} & \text{if } 3 \le t < 4 \\ 1 & \text{if } t \ge 4 \end{cases}$$

(b) Y is discrete.

(c)

$$f_Y(j) = \begin{cases} \frac{3}{10} & \text{if } j = -1 \\ \frac{6}{10} & \text{if } j = 3 \\ \frac{1}{10} & \text{if } j = 4 \end{cases}$$

(d)

$$\mathbb{E}[Y] = (-3)\frac{1}{10} + 3\frac{6}{10} + 4\frac{1}{10} = \frac{19}{10}.$$

2. (a)

$$\mathbb{P}\{Y \ge X\} = \int_{s=0}^{\infty}\int_{t=s}^{\infty} \lambda e^{-\lambda s}\nu e^{-\nu t} dt\, ds = \int_{s=0}^{\infty} \lambda e^{-\lambda s - \nu s} ds = \frac{\lambda}{\lambda+\nu}.$$

(b)

$$\mathbb{P}\{Z > 5\} = \mathbb{P}\{X > 5\}\mathbb{P}\{Y > 5\} = \left(\int_{s=5}^{\infty} \lambda e^{-\lambda s} ds\right)\left(\int_{t=5}^{\infty} \nu e^{-\nu t} dt\right) = e^{-5(\lambda+\nu)}.$$

(c)

$$\mathbb{P}\{Z \le 5\} = 1 - e^{-5(\lambda+\nu)}.$$

(d)

$$F_Z(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-(\lambda+\nu)t} & \text{if } t \ge 0 \end{cases}$$

3. (a)

$$f_X(t) = \begin{cases} 6t(1-t) & \text{if } t \in [0,1] \\ 0 & \text{else} \end{cases}$$

(b)

$$\mathbb{P}\left\{Z \le \frac{1}{3}\right\} = \int_{s=0}^{1/3}\int_{t=0}^{1/3-s} (6t) dt\, ds = 3\int_{s=0}^{1/3}\left(\frac{1}{3} - s\right)^2 ds = \left(\frac{1}{3}\right)^3.$$

(c)

$$F_Z(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^3 & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(d)

$$f_Z(t) = \begin{cases} 3t^2 & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$