

Math 361, Section X1, Fall 2002
Final, December 16

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 150 Points

1. 30 points Suppose that there are 10 married couples in a room. Suppose for specificity that one of the married couples is Don and Donna. Randomly pick four people to sit at a table.
 - (a) 10 points What is the probability that you pick Don and Donna?
 - (b) 10 points What is the probability that you pick Don and Donna and another married couple?
 - (c) 10 points What is the probability that you pick two married couples?
2. 20 points Suppose that X and Y have joint density

		Y			
		0	1	2	3
	-1	3/27	1/27	1/27	3/27
X	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

In other words, $f_{X,Y}(0, 1) = 0$.

- (a) 5 points Compute f_X .
- (b) 10 points Compute $\mathbb{E}[X^2Y]$.

Now define $Z \stackrel{\text{def}}{=} X + Y$.

- (c) 5 points Compute $\mathbb{P}\{Z = 1\}$.

3. 40 points Fix $\lambda > 0$. Let X and Y be continuous random variables with joint density

$$f_{X,Y}(s, t) = \begin{cases} \lambda e^{-\lambda(t-s)} & \text{if } t \geq s \text{ and } 0 \leq s \leq 1 \\ 0 & \text{else} \end{cases}$$

- (a) 10 points Compute $f_Y(4)$
- (b) 10 points Compute $f_Y(\frac{1}{3})$
- (c) 10 points Compute f_Y .
- (d) 10 points Compute $f_{X|Y}(s|4)$ for $0 < s < 1$.

4. 50 points Let X be a random variable with cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 - e^{-3t} & \text{if } 1 \leq t < 10 \\ 1 & \text{if } t \geq 10 \end{cases}$$

- (a) 5 points Compute $\mathbb{P}\{X < 10\}$.
(b) 5 points Compute $\mathbb{P}\{X = 1\}$.

Define now

$$\tilde{F}(t) \stackrel{\text{def}}{=} \mathbb{P}\{X \leq t | X \in (1, 10)\}$$

for all $t \in \mathbb{R}$ (this is the cumulative distribution of X given that $X \in (1, 10)$).

- (c) 5 points Compute $\mathbb{P}\{X \in (1, 10)\}$
(d) 5 points Compute $\mathbb{P}\{X \leq 5 \text{ and } X \in (1, 10)\}$.
(e) 3 points Compute $\tilde{F}(5)$.
(f) 5 points Compute $\mathbb{P}\{X \leq 12 \text{ and } X \in (1, 10)\}$.
(g) 3 points Compute $\tilde{F}(12)$.
(h) 5 points Compute $\mathbb{P}\{X \leq -2 \text{ and } X \in (1, 10)\}$.
(i) 3 points Compute $\tilde{F}(-2)$.
(j) 5 points Compute $\mathbb{P}\{X \leq 1 \text{ and } X \in (1, 10)\}$.
(k) 3 points Compute $\tilde{F}(1)$.
(l) 3 points Compute $\tilde{F}(t)$ for all t .
5. 10 points Let X and Y be independent Gaussian random variables with mean 0 and variance 1; i.e.,

$$f_X(t) = f_Y(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] \quad t \in \mathbb{R}$$

Define $Z_1 \stackrel{\text{def}}{=} 7X + 3$ and $Z_2 \stackrel{\text{def}}{=} 5X + 4Y$.

- (a) 5 points Compute the moment generating function of Z_1 .
(b) 5 points Compute the moment generating function of Z_2 .

ANSWERS

1. (a) $\frac{\binom{18}{2}}{\binom{20}{4}}$.
 (b) $\frac{9}{\binom{20}{4}}$.
 (c) $\frac{\binom{10}{2}}{\binom{20}{4}}$.

2. (a)

$$f_X(j) = \begin{cases} \frac{8}{27} & \text{if } j = -1 \\ \frac{12}{27} & \text{if } j = 0 \\ \frac{7}{27} & \text{if } j = 3 \end{cases}$$

- (b)

$$\mathbb{E}[X^2Y] = (-1)^2(1)\frac{1}{27} + (-1)^2(2)\frac{1}{27} + (-1)^2(3)\frac{3}{27} + (3)^2(3)\frac{4}{27} = \frac{120}{27}.$$

(c) $\mathbb{P}\{Z = 1\} = f_{X,Y}(-1, 2) + f_{X,Y}(0, 1) = \frac{1}{27}.$

3. (a) $f_Y(4) = \int_{s=0}^1 \lambda e^{-\lambda(4-s)} ds = e^{-3\lambda} - e^{-4\lambda}.$

(b) $f_Y(\frac{1}{3}) = \int_{s=0}^{1/3} \lambda e^{-\lambda((1/3)-s)} ds = 1 - e^{-(1/3)\lambda}.$

- (c)

$$f_Y(t) = \begin{cases} e^{-\lambda(t-1)} - e^{-\lambda t} & \text{if } t \geq 1 \\ 1 - e^{-\lambda t} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

- (d)

$$f_{X|Y}(s|4) = \frac{f_{X,Y}(s, 4)}{f_Y(4)} = \frac{\lambda e^{-\lambda(4-s)}}{e^{-3\lambda} - e^{-4\lambda}}.$$

4. (a) $\mathbb{P}\{X < 10\} = 1 - e^{-30}.$

(b) $\mathbb{P}\{X = 1\} = 1 - e^{-3}.$

(c) $\mathbb{P}\{X \in (1, 10)\} = F_X(10-) - F_X(1) = e^{-3} - e^{-30}.$

(d) $\mathbb{P}\{X \leq 5 \text{ and } X \in (1, 10)\} = \mathbb{P}\{X \in (1, 5]\} = F_X(5) - F_X(1) = e^{-3} - e^{-15}.$

(e) $\tilde{F}(5) = (e^{-3} - e^{-15}) / (e^{-3} - e^{-30}).$

(f) $\mathbb{P}\{X \leq 12 \text{ and } X \in (1, 10)\} = \mathbb{P}\{X \in (1, 10)\} = e^{-3} - e^{-30}.$

(g) $\tilde{F}(12) = 1.$

(h) $\mathbb{P}\{X \leq -2 \text{ and } X \in (1, 10)\} = \mathbb{P}(\emptyset) = 0.$

(i) $\tilde{F}(-2) = 0.$

(j) $\mathbb{P}\{X \leq 1 \text{ and } X \in (1, 10)\} = \mathbb{P}(\emptyset) = 0.$

(k) $\tilde{F}(1) = 0.$

(1)

$$\tilde{F}(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{e^{-3} - e^{-3t}}{e^{-3} - e^{-30}} & \text{if } 1 \leq t < 10 \\ 1 & \text{if } t \geq 10 \end{cases}$$

5. (a) $\mathbb{E}[e^{\theta Z_1}] = \exp\left[\frac{49}{2}\theta^2 + 3\theta\right]$.
(b) $\mathbb{E}[e^{\theta Z_2}] = \exp\left[\frac{41}{2}\theta^2\right]$.