Math 451, Spring 2004 **Homework 2**
Richard Sowers **due February 11**

1. $\boxed{\text{10 points}}$ Fix $\rho \in (0,1)$. Let X_n be geometric with parameter ρ/n. Let $Y_n \stackrel{\text{def}}{=} \frac{1}{n}X_n$ and let μ_n be the law of Y_n. Directly show that for bounded $f \in C(\mathbb{R})$,

$$\lim_{n \nearrow \infty} \int_{z \in \mathbb{R}} f(z)\mu_n(dz) = \int_{z=0}^{\infty} f(z)\rho e^{-\rho z} dz.$$

2. $\boxed{\text{10 points}}$ Show that convergence in probability implies weak convergence. In other words, let $(X_n;\, n \in N)$ be a sequence of real-valued random variables which converges in probability to another random variable X. Let μ_n be the law of X_n, and let μ be the law of X. Show that μ_n tends weakly to μ.

3. $\boxed{\text{40 points}}$ For convenience, we will let $\mathscr{P}(\mathbb{R})$ be the collection of probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. For any subset A of $\mathbb{R}$ and any $\delta > 0$, define

$$A^\delta \stackrel{\text{def}}{=} \{x \in \mathbb{R} : |x-y| \le \delta \text{ for some } y \in A\}$$

Fix μ and ν in $\mathscr{P}(\mathbb{R})$, and define

$$\rho(\mu,\nu) \stackrel{\text{def}}{=} \inf\left\{\delta > 0 : \mu(F) \le \nu(F^\delta) + \delta \text{ for all closed subsets } F \text{ of } \mathbb{R}\right\}.$$

Fix now sequence μ and a sequence $(\mu_n;\, n \in \mathbb{N})$ in $\mathscr{P}(\mathbb{R})$.

   (a) $\boxed{\text{10 points}}$ Suppose that μ_n is the uniform law on $[0, 1/n]$, and suppose that μ is a Dirac measure at 0. Show that $\lim_{n\nearrow\infty} \rho(\mu_n, \mu) = 0$.

   (b) $\boxed{\text{10 points}}$ Suppose that $\lim_{n\nearrow\infty} \rho(\mu_n, \mu) = 0$. Show that $\overline{\lim}_{n\nearrow\infty} \mu_n(F) \le \mu(F)$ for all closed subsets F of $\mathbb{R}$.

   (c) $\boxed{\text{10 points}}$ Suppose that $\overline{\lim}_{n\nearrow\infty} \mu_n(F) \le \mu(F)$ for all closed subsets F of $\mathbb{R}$. Show that $\lim_{n\nearrow\infty} \rho(\mu_n, \mu) = 0$.

   (d) $\boxed{\text{10 points}}$ Suppose that $\rho(\mu, \nu) = 0$. Show that $\mu = \nu$.