Math 451, Spring 2004 **Homework 5**
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1. $\boxed{\text{10 points}}$ Let X be a submartingale with respect to a filtration $\mathscr{F}$. Let τ be a bounded stopping time, and define

$$X^\tau_n \overset{\text{def}}{=} X_{\tau\wedge n}$$

Show that X^τ is also a submartingale with respect to $\mathscr{F}$.

2. $\boxed{\text{10 points}}$ (Exercise 5.14 in book). Suppose that τ is a stopping time with $\mathbb{P}\{\tau < \infty\} = 1$. Suppose also that $\tau_n \nearrow \tau$, where the τ_n's are also stopping times. Show that

$$\mathscr{F}_\tau = \bigvee_n \mathscr{F}_{\tau_n}.$$

Answers

1. $\boxed{\text{10 points}}$ Fix n First note that the $\{\tau \le n\}$ and thus $\{\tau > n\}$ are in $\mathscr{F}_n$ and that the random variablble $X_{\tau\wedge n}$ is $\mathscr{F}_n$-measurable. Thus

$$\begin{aligned}\mathbb{E}[X^{\tau}_{n+1}|\mathscr{F}_n] = \mathbb{E}[X_{\tau\wedge(n+1)}|\mathscr{F}_n] &= \mathbb{E}[X_{\tau\wedge(n+1)}\chi_{\{\tau>n\}}|\mathscr{F}_n] + \mathbb{E}[X_{\tau\wedge(n+1)}\chi_{\{\tau\le n\}}|\mathscr{F}_n] \\ &= \mathbb{E}[X_{n+1}\chi_{\{\tau>n\}}|\mathscr{F}_n] + \mathbb{E}[X_{\tau\wedge n}\chi_{\{\tau\le n\}}|\mathscr{F}_n] \\ &= \mathbb{E}[X_{n+1}|\mathscr{F}_n]\chi_{\{\tau>n\}} + X_{\tau\wedge n}\chi_{\{\tau\le n\}} \ge X_n\chi_{\{\tau>n\}} + X_{\tau\wedge n}\chi_{\{\tau\le n\}} \ge X_{\tau\wedge n}.\end{aligned}$$

2. $\boxed{\text{10 points}}$ Since $\tau_n \le \tau$, $\mathscr{F}_{\tau_n} \subset \mathscr{F}_\tau$ for every n, so

$$\bigvee_n \mathscr{F}_{\tau_n} \subset \mathscr{F}_\tau.$$

Fix $A \in \mathscr{F}_\tau$. Then

$$A = \bigcup_{n=1}^{\infty} A \cap \{\tau = n\}$$

and, since τ and the τ_j's take on discrete values,

$$A \cap \{\tau = n\} = \bigcup_{j=1}^{\infty} A \cap \{\tau = n\} \cap \{\tau_j = n\}.$$

Note that $A \cap \{\tau = n\} = (A \cap \{\tau \le n\}) \setminus \{\tau \le n-1\} in \mathscr{F}_n$. If $n > k$, then

$$A \cap \{\tau = n\} \cap \{\tau_j = n\} \cap \{\tau_j \le k\} = \emptyset \in \mathscr{F}_k$$

and if $n \le k$, then

$$A \cap \{\tau = n\} \cap \{\tau_j = n\} \cap \{\tau_j \le k\} A \cap \{\tau = n\} \cap \{\tau_j = n\} \in \mathscr{F}_n \subset \mathscr{F}_k$$

Thus $A \cap \{\tau = n\} \cap \{\tau_j = n\} \in \mathscr{F}_{\tau_j}$.