1. [10 points] Show that for any $\varphi \in C^1([0,1])$ (i.e., φ is continuous and has a continuous first derivative on $[0,1]$),

$$\int_0^1 |\dot{\varphi}(t)|^2 dt = \sup_{0=t_0 \le t_1 \dots t_N = 1} \sum_{i=0}^{N-1} \frac{|\varphi(t_{i+1}) - \varphi(t_i)|^2}{t_{i+1} - t_i}.$$

2. [10 points] Consider the 1-dimensional SDE

$$dX_t = b(X_t)dt + dW_t$$

where

$$\inf_{x \ne 0} \frac{-b(x)}{x} > 0.$$

Compute the stationary distribution, if one exists, of this SDE.

3. [10 points] Fix a Brownian motion W. Fix also a smooth function G such that

$$\inf_{x \in \mathbb{R}} G(x) > 0.$$

Suppose that we have the solution of the ODE

$$\begin{aligned} \dot{\tau}(t) &= G(W_{\tau(t)}) \qquad t > 0 \\ \tau(0) &= 0. \end{aligned}$$

Define then

$$X_t \stackrel{\text{def}}{=} W_{\tau(t)} \qquad t > 0$$

I claim that then

$$dX_t = b(X_t)dt + \sigma(X_t)dV_t$$

for SOME Brownian motion V. Compute b and σ. Hint: You may want to use the martingale problem.

4. [10 points] Let p be the solution of the Zakai equation for the filtering problem for

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t \\ dY_t &= h(X_t)dt + dV_t \end{aligned}$$

Define the normalized density

$$\hat{p}(t,x) \stackrel{\text{def}}{=} \frac{p(t,x)}{\int_{z \in \mathbb{R}} p(t,z)dz}.$$

Find the stochastic PDE for $\hat{p}$.

5. **10 points** Consider the 2-dimensional SDE

$$\begin{aligned} d\theta_t^\varepsilon &= \frac{1}{\varepsilon^2}\omega(Z_t^\varepsilon)dt \\ dZ_t^\varepsilon &= \sigma(\theta_t^\varepsilon, Z_t^\varepsilon)dW_t \\ \theta_0^\varepsilon &= 0 \qquad \text{and} \qquad Z_0^\varepsilon = z_\circ \end{aligned}$$

for some $z_\circ \in \mathbb{R}$, and where W is a standard Brownian motion. We assume that ω and σ are bounded functions and Lipshitz-continuous functions. Show that the law of Z^ε is tight; i.e., show that for each $T > 0$ and $\mu > 0$,

$$\lim_{\eta \to 0} \sup_{\varepsilon \in (0,1)} \mathbb{P}\left\{ \sup_{\substack{0 \le s < t \le T \\ |t-s| < \eta}} |Z_t^\varepsilon - Z_s^\varepsilon| \ge \mu \right\} = 0.$$