**Math 595, Fall 2005** **Homework 3**
**Richard Sowers** **due February 28**

Fix $\mu \in \mathscr{P}(\mathbb{R}^d)$, and define

$$\Lambda(\theta) \overset{\text{def}}{=} \ln \int_{z\in\mathbb{R}^d} \exp\left[\langle \theta, z\rangle\right] \mu(dz)$$
$$I(z) \overset{\text{def}}{=} \sup_{\theta\in\mathbb{R}^d} \{\langle \theta, z\rangle - \Lambda(\theta)\}$$

1. [10 points] [Lower Semicontinuity] Consider a function $f : \mathsf{X} \to \mathbb{R}$, where $\mathsf{X}$ is a Polish space. Show that the following two properties are equivalent:
   i) for each $s \in \mathbb{R}$, $\{x \in \mathsf{X} : f(x) \le s\}$ is closed.
   ii) if $\{x_n;\, n \in \mathbb{N}\}$ is a sequence in $\mathsf{X}$ converging to $x^* \in \mathsf{X}$, then we have that $f(x^*) \le \underline{\lim}_{n\to\infty} f(x_n)$.

2. [20 points] Consider $\Lambda$ as above.
   (a) [10 points] Show that $\Lambda$ is convex.
   (b) [10 points] Show that $I \ge 0$.

3. [30 points] [Moderate Deviations] Assume that $d = 1$ and that $\Lambda$ is finite and differentiable in a neighborhood of the origin. Let $\{\xi_1, \xi_2 \dots\}$ be an i.i.d. collection of random variables with common law $\mu$. Assume that $\int z\mu(dz) = 0$. Define $S_n \overset{\text{def}}{=} \sum_{i=1}^n \xi_i$ for all $n \in \mathbb{N}$. Fix $\alpha \in (1/2, 1)$.
   (a) [10 points] Show that $\lim_{n\to\infty} \frac{S_n}{n^\alpha} = 0$ in probability.
   (b) [20 points] We next want to guess at a large deviations principle for $S_n/n^\alpha$. Find $\beta > 0$ such that
   $$\tilde{\Lambda}_\beta(\theta) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n^\beta} \ln \mathbb{E}\left[\exp\left[n^\beta \theta \left(\frac{S_n}{n^\alpha}\right)\right]\right]$$
   is nontrivial. In this case, compute $\tilde{\Lambda}_\beta$ (hint: keep in mind the possibility of differentiating $\Lambda$ at the origin).

4. [10 points] Suppose that $d = 2$. Define $\psi(x_1, x_2) \overset{\text{def}}{=} x_1$ for all $(x_1, x_2) \in \mathbb{R}^2$. Assume that $S_n/n$ has a large deviations principle (in $\mathbb{R}^2$) with rate function $I$. Cramer's theorem should also give us a rate function for the first component of $S_n/n$. Show that the contraction principle gives the same rate function. You may switch an $\inf\sup$ to a $\sup\inf$.

5. [10 points] Suppose that $X^\varepsilon$ satisfies the SDE
$$dX^\varepsilon_t = -\lambda X^\varepsilon_t dt + \varepsilon dW_t$$
$$X^\varepsilon_0 = 0$$

(although it doesn't matter, let's assume that $\lambda > 0$). Note that this can be rewritten as

$$X_t^\varepsilon = \varepsilon W_t - \lambda \int_{s=0}^t e^{\lambda(t-s)}\{\varepsilon W_s\} ds.$$

Thus $X^\varepsilon$ is a transformation of the process $Y_t^\varepsilon \overset{\text{def}}{=} \varepsilon W_t$. Use the LDP for $Y^\varepsilon$ and the contraction principle to get an LDP for $X^\varepsilon$. Note: I really want you to use the contraction principle. The process $X^\varepsilon$ is Gaussian, so you can use other methods. I want the contraction principle.