

Math 531. Analytic Theory of Numbers I

Textbook: T.M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, 1995.

1. Arithmetic functions
 - Additive and multiplicative arithmetic functions
 - Definitions and basic properties of the functions $\mu(n)$, $\phi(n)$, $\sigma(n)$, $\omega(n)$, $\Omega(n)$, $\Lambda(n)$.
 - Dirichlet multiplication and Möbius inversion
 - Averages of arithmetic functions; estimates for sums like $\sum_{n \leq x} \mu^2(n)$, $\sum_{n \leq x} \phi(n) / n$, with error terms of the form $O(x^a)$ (or better)
 - Estimation of $\sum_{n \leq x} d(n)$ by the Dirichlet hyperbola method
 - Partial summation
 - Euler summation formula
2. Elementary theorems on the distributions of primes
 - Definition of $\pi(x)$, $\varphi(x)$, $\psi(x)$, and relations between these functions
 - Chebyshev's estimates
 - Mertens's type formulae (i.e., estimates for $\prod_{p \leq x} (1 - 1/p)$, $\prod_{p \leq x} 1/p$, $\prod_{p \leq x} (\log p) / p$, and similar products)
 - Equivalence of the prime number theorem and the estimate $\sum_{n \leq x} \mu(n) = o(x)$
3. Dirichlet series and Euler products
 - Regions for convergence, absolute convergence, and uniform convergence of a Dirichlet series; abscissa of convergence and abscissa of absolute convergence
 - Analyticity of a Dirichlet series
 - Connections between growth of a summatory functions $\sum_{n \leq x} f(n)$ and abscissa of convergence of the associated Dirichlet series.
 - Relation between convolution of arithmetic functions and Dirichlet series.
 - Euler product formula and applications (e.g., to proving relations between arithmetic functions)
4. Properties of the Riemann zeta function
 - General analytic properties of $\zeta(s)$ (poles, zeros, analytic continuation)
 - Bounds for $\zeta(s)$, $\zeta'(s)$ and $1 / \zeta(s)$ in a region $s \geq 1 - c / \log(|t| + 2)$
 - Functional equation of the zeta function
 - Properties of the Gamma function
5. Analytic proof of the prime number theorem (PNT)
 - Perron's formula
 - Proof of the PNT with error term $O(x (\log x)^{-A})$ or better (outline)
6. Dirichlet's theorem on primes in arithmetic progressions
 - Definition of characters and Dirichlet L-functions
 - Orthogonality relation for characters
 - Non-vanishing of $L(1, \chi)$
 - Proof of Dirichlet's theorem (outline)