Recent results on Lieb-Thirring inequalities

Abstract: Lieb-Thirring inequalities are bounds on moments of the negative eigenvalues of a Schrödinger operator. They are of the form

$$\mathrm{tr}[(H)_-^\gamma] = \sum_n |E_n|^\gamma \leq \frac{C_{\gamma,d}}{(2\pi)^d} \iint (P^2 + V(X))_-^\gamma \, dPdX$$

where $H = -\Delta + V$ is the one-particle Schrödinger operator on $L^2(\mathbb{R}^d)$, $E_1 < E_2 \leq \ldots \leq 0$ its negative eigenvalues, $P^2 + V(X)$ the classical symbol on the phase space $\mathbb{R}^{2d}$, and the bound can only hold for suitable γ depending on the dimension d, e.g., $\gamma \geq 0$ for $d \geq 3$.

The motivation to study and prove these type of bounds originates in physics. In fact, one of the most important cases used by Lieb and Thirring, $\gamma = 1$ and $d = 3$, was instrumental in their proof of stability of matter since it gives a lower bound for the kinetic energy of fermions in term of the Thomas-Fermi kinetic energy. It is well-known that $C_{\gamma,d} \geq 1$. If $C_{1,3} = 1$, then the Lieb-Thirring inequality gives exactly the kinetic energy bound predicted by Thomas and Fermi. Another important case is $\gamma = 0$, $d \geq 3$, the Cwikel-Lieb-Rozenblum bound on the number of negative eigenvalues.

Besides these applications in physics, the Lieb-Thirring bound is very interesting even from a purely mathematical point of view; it bounds a quantum mechanical property, the discrete eigenvalues of an operator, in terms of its classical symbol.

We will discuss some of the recent results on Lieb-Thirring inequalities. Especially sharp bounds in one dimension and improved bounds on $C_{\gamma,d}$ in all dimensions. It turns out that a lot of results in high dimension can be deduced from lower dimensions, which at first sight might come as a total surprise. In particular, this approach gave bounds on $C_{\gamma,d}$, which are uniform in d or even best possible. For example, Ari Laptev and Timo Weidl showed that $C_{\gamma,d} = 1$ for $\gamma \geq 3/2$ all d and Laptev, Weidl, and the speaker showed that $C_{\gamma,d} \leq 2$ for $1 \leq \gamma < 3/2$ and all d. The basic input for this last bound is a sharp bound in one-dimension for $\gamma = 1/2$ due to Lieb, Thomas, and the speaker.