Differential Geometry Comprehensive Exam
August 2006.

1. Let $M$ be the set of (straight, nondegenerate) circles in the 2-sphere $S^2 \subseteq \mathbb{R}^3$. Describe a natural topology on $M$. Show that $M$ is a manifold. What is the dimension of $M$? Is $M$ orientable?

2. Is there an immersion of the 2-sphere $S^2$ to $S^1 \times \mathbb{R}$?

3. Give a definition of the de Rham cohomology, $H^p_{dR}(M)$, of a smooth manifold $M$. Prove that if $M$ is a compact orientable manifold (without boundary) of dimension $n$, then $H^n_{dR}(M) \neq 0$.

4. (a) Give a definition of a connection on a vector bundle over a manifold $M$.
(b) Let $M$ be a smooth manifold. Is there a connection $\nabla$ on $TM$ such that $\nabla_X Y = \nabla_Y X$ for all vector fields $X$ and $Y$ on $M$?
(c) Is there a connection on $TM$ such that $\nabla_X \nabla_Y Z = \nabla_Y \nabla_X Z$ for all vector fields $X$, $Y$ and $Z$ on $M$?