MATH 550 COMPREHENSIVE EXAMINATION
August 2006

ORDINARY DIFFERENTIAL EQUATIONS AND DISCRETE MAPS
DO ALL PROBLEMS.

1. Area preserving map
Consider the following map \( f \) of the 2-torus \( T^2 \) defined by
\[
 f(x, y) = (x + y, x + 2y) \mod 1 \quad \text{for} \ 0 \leq x, y < 1.
\]

a) Show that \( f \) is area-preserving.
b) Show that \( f \) has a countable infinity of periodic points.
c) Show that the periodic points of \( f \) are dense in \( T^2 \).

2. Hamiltonian systems and area-preserving flows
Consider the following system of first order ordinary differential equations:
\[
dx/dt = f(x, y) \quad \text{and} \quad dy/dt = g(x, y)
\]
where \( f \) and \( g \) are of class \( C^\infty \) and satisfy \( \partial f/\partial x + \partial g/\partial y = 0 \) for all \((x, y)\) in the plane.

a. Show that the local flow of the vector field \((f, g)\) is area-preserving.
b. Show that there is a \( C^\infty \) real valued function \( F \) (the Hamiltonian) such that
\[
f = \partial F/\partial y \quad \text{and} \quad g = - \partial F/\partial x.
\]
c. If \( f \) and \( g \) are defined by \( f(x, y) = 3x^2y^2 \) and \( g(x, y) = -2xy^3 - 3 \), find \( F(x, y) \).

3. Let \( f(x) = x^{2/3} \) for \( x \neq 0 \) and \( f(0) = 0 \).
   a. Decide whether the initial value problem \( \frac{dx}{dt} = f(x), \ x(0) = 0 \) has a unique solution.
   b. Can any solution of the initial value problem with \( x(0) \neq 0 \) become unbounded in \( x \) as \( t \) increases?
   c. Are solutions of the initial value problem with \( x(0) \neq 0 \) defined for all time?

4. Let \( f(x) = x^{2/3} \sin (1/x) \) for \( x \neq 0 \) and \( f(0) = 0 \).
   a. Decide whether the initial value problem \( \frac{dx}{dt} = f(x), \ x(0) = 0 \) has a unique solution. If the solution is unique, then prove it. If not, prove it.
   b. What conditions does \( f \) satisfy in a neighborhood of the origin?
   c. Can any solution of the initial value problem with \( x(0) \neq 0 \) become unbounded in \( x \) as \( t \) increases?
   d. Are solutions of the initial value problem with \( x(0) \neq 0 \) defined for all time?