COMBINATORICS COMPREHENSIVE - Fall 2006

Do FIVE problems from Part I and THREE from Part II. Passing requires good performance on each Part. Justify answers; give clear statements of any theorems you use.

Part I

1. Let \( a_n \) be the number of arrangements of \( 2n \) people in 2 rows of length \( n \) such that heights increase in each row and column. By establishing a bijection involving a set of known size, prove that \( a_n = \frac{1}{n+1} \binom{2n}{n} \).

2. Let \( a(d_1, \ldots, d_n) \) be the number of trees with vertex set \([n]\) in which for each \( i \), the degree of \( i \) is \( d_i \). Obtain a recurrence for \( a(d_1, \ldots, d_n) \) and use it to prove that \( a(d_1, \ldots, d_n) = \binom{n-2}{d_1-1, \ldots, d_n-1} \). Use this result to prove Cayley's Formula for the number of trees with vertex set \([n]\).

3. Use generating functions to evaluate the sum below. (Hint: It is easier without convolution.)

\[
\sum_{k=1}^{n} (-1)^{n-k} k \binom{n}{k} 2^k.
\]

4. A rotating square table has a pocket at each corner. In each pocket we have the choice to place 1, 2, or 3 marbles. Compute the total number of distinguishable arrangements. Explain how to use the pattern inventory to obtain the number of distinguishable arrangements with a total of 7 marbles.

5. Use Tutte's 1-factor Theorem to prove that every connected line graph of even order has a perfect matching. Interpret the result as a statement about decomposition of graphs into subgraphs.

6. Let \( G \) be a \( k \)-regular graph with connectivity 1. Determine \( \chi'(G) \).

7. Let \( G \) be a 3-regular connected plane graph in which every vertex is incident to one face of length 4, one face of length 6, and one face of length 8. Without drawing \( G \), count the faces of \( G \).

Part II

8. Prove that every simple Eulerian graph with at least three vertices has at least three vertices with the same degree.

9. Consider a red/blue-coloring of the edges of a complete graph with more than \( m^2 \) vertices. Suppose that the red graph is transitively orientable. Prove that the coloring has a monochromatic complete subgraph of order \( m + 1 \). (Hint: Use posets.)

10. Let \( G \) be a graph with \( m \) edges. Use the probabilistic method to prove that if \( G \) has a matching of size \( k \), then \( G \) has a bipartite subgraph with at least \( (m + k)/2 \) edges.

11. Consider the random graph model \( G(n, p) \), where \( p = o(1/n) \). Prove that in this model almost every graph has no cycles.