Comprehensive Exam in PDE's - August 2007

Problem 1:

Let \( G(\rho) = \rho(1 - \rho/\rho_{\text{max}}) \), where \( \rho_{\text{max}} \) is a given constant. Using the method of characteristics for quasilinear first order equations, solve

\[
G(\rho)_x + \rho_y = 0 \quad \text{for } x \in \mathbb{R}, 0 < y < 2,
\]

given initial data

\[
\rho(x, 0) = \begin{cases} 
\rho_{\text{max}}/2 & \text{for } -1 < x < 0, \\
0 & \text{otherwise.}
\end{cases}
\]

A good way to present the solution is to sketch the projected characteristics. Be sure to show any shock curves or rarefaction fans, with justification. You are not required to find a formula for the whole solution, though.

(ii) What happens at \( y = 2 \) that changes the solution? (Give a one sentence answer.)
Problem 2 (25 points)

Suppose $u$ is harmonic and non-negative on $\mathbb{R}^n$.

i) Show, by using the Poisson formula for the ball, that for each $\zeta > 0$,

$$\frac{R^{n-2}(R - |\zeta|)}{(R + |\zeta|)^{n-1}}u(0) \leq u(\zeta) \leq \frac{R^{n-2}(R + |\zeta|)}{(R - |\zeta|)^{n-1}}u(0) \quad \text{for } |\zeta| < R.$$

ii) Show that if $u$ is harmonic and non-negative on $\mathbb{R}^n$ then $u$ must be constant.

iii) Use (ii) to deduce the following slight generalization of Liouville's theorem: If $v : \mathbb{R}^n \to \mathbb{R}$ is harmonic and either bounded from below or from above, then $v$ is constant.

**Hint:** The Poisson kernel is given by

$$K(x, \xi) = \frac{R^2 - |\xi|^2}{R \omega_n |x - \xi|^n}$$

where $\omega_n$ is the area of the $(n-1)$ sphere.
Problem 3. An Inverse Wave Problem
Let \( G(x, t) = \frac{1}{2} \chi_{[-t,t]}(x) H(t) \). Recall that \( \chi_A \), the characteristic function of a set \( A \) is defined to be

\[
\chi_A(x) = \begin{cases} 
1 & x \in A \\
0 & x \notin A
\end{cases}
\]

and the Heaviside function \( H(t) \) to be

\[
H(t) = \begin{cases} 
1 & t > 0 \\
0 & t \leq 0
\end{cases}
\]

(i) Show that (in the sense of distributions) \( G(x, t) \) solves

\[
G_{tt} - G_{xx} = \delta(x) \delta(t)
\]

Be sure to state clearly what it means for \( G \) to satisfy the above equation in the sense of distributions. You will find it easier to work in characteristic coordinates.

(ii) Show that

\[
U(x, t) = \int G(x - x', t) g(x') dx'
\]

is a classical solution of

\[
U_{tt} - U_{xx} = 0
\]

if \( g \) is \( C^1 \). What initial conditions does it satisfy?

(iii) Using (i) and (ii) derive the D'Alembert solution to the one dimensional wave equation on the whole line for general initial data.
Problem 4: Maximum Principle

(i) Suppose that \( u(x,t) \) satisfies a linear PDE with vanishing boundary conditions, and that this evolution is NOT positivity preserving: there exists initial data \( f(x) = u(x,0) \geq 0 \) such that \( u(x,t) < 0 \). Show that the linear PDE does not satisfy the maximum principle.

(ii) Solve the equation

\[
 u_t = \left( \frac{\partial^{2008} u}{\partial x^{2008}} + \frac{\partial^{2008} u}{\partial y^{2008}} \right) \quad u(x,y,0) = g(x,y)
\]

via Fourier transform. Write your solution in the form of a convolution of the initial data \( g(x,y) \) with a kernel \( K(x,y,t) \).

(iii) Show that a necessary condition for the above equation to be positivity preserving is that the kernel \( K(x,y,t) \) be strictly positive.

(iv) Show that the kernel \( K(x,y,t) \) is not a strictly positive function and conclude that the evolution above does not have a maximum principle.

Hint: Consider \( \int x^k K(x,y,t) dx dy \) for appropriate \( k \). What does this tell you about the kernel?