COMBINATORICS COMPREHENSIVE - Fall 2007

Submit only FIVE problems from Part I and THREE problems from Part II; 80 points possible. Passing requires good performance on both Parts. Justify answers; GIVE CLEAR STATEMENTS of any theorems you use.

Part I

1. By counting a set in two ways, prove that \( \sum_{i=1}^{n} i(n - i) = \sum_{i=1}^{n} \binom{i}{2} \).

2. Let \( \{a\} \) satisfy \( a_n = 3a_{n-1} - 2a_{n-2} + 2^n \) for \( n \geq 2 \), with \( a_0 = a_1 = 1 \). Express the generating function for \( \{a\} \) as a ratio of two polynomials. Obtain a formula for \( a_n \) as a function of \( n \).

3. Use generating functions to evaluate the sum \( \sum_{k=0}^{r} (-1)^k \binom{n}{k} \binom{n}{r-k} \).

4. Suppose we roll a six-sided die until each of the numbers one through five have appeared at least once. What is the probability that we succeed sometime during the first \( n \) rolls?

5. Let \( S \) be a set of permutations of \([n]\). Prove that if \( |S| \leq n/2 \), then some permutation of \([n]\) differs in every position from every member of \( S \). (Hint: Model this using a graph problem.)

6. Let \( G \) be a \( k \)-regular graph with connectivity 1. Determine \( \chi'(G) \).

7. Use Euler's Formula to count the regions determined by a configuration of \( n \) lines in the plane when every two lines cross but no three lines have a common point.

Part II

8. Prove that \( 2m - 1 \) is the minimum \( t \) such that every 2-coloring of \( E(K_{t,t}) \) has a monochromatic connected subgraph with \( 2m \) vertices.

9. Consider a red/blue-coloring of the edges of a complete graph with more than \( m^2 \) vertices. Suppose that the red graph has a transitive orientation. Prove that the coloring has a monochromatic complete subgraph of order \( m + 1 \). (Hint: Use posets.)

10. Let \( G \) be a bipartite graph with \( n \) vertices in which every vertex is given a list of more than \( \log_2 n \) usable colors. Prove that a proper coloring of \( G \) can be chosen from the lists.

11. Suppose that a \( (4m - 1, 2m - 1, m - 1) \)-design exists. Prove that there is a Hadamard matrix of order \( 4m \).