Justify your answers. Good luck.

**Problem 1.** Let \( p \) be a prime number.
- (10 points) Define a Sylow \( p \)-subgroup of a finite group.
- (10 points) Let \( \mathbb{F}_p \) be a finite field with \( p \) elements, \( GL(2, \mathbb{F}_p) \) the group of invertible 2x2 matrices with \( \mathbb{F}_p \)-coefficients, and \( U(2, \mathbb{F}_p) = \{ \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p \} \) the subgroup of upper-triangular unipotent matrices. Show that \( U(2, \mathbb{F}_p) \) is a Sylow \( p \)-subgroup of \( GL(2, \mathbb{F}_p) \).
- (10 points) Find the number of Sylow \( p \)-subgroups of \( GL(2, \mathbb{F}_p) \).

**Problem 2.** (10 points) Find (a) all homomorphisms of additive groups \( \mathbb{Z} \to \mathbb{Q} \) and (b) all homomorphisms of rings \( \mathbb{Z} \to \mathbb{Q} \).

[In this problem ring homomorphisms are not required to map 1 to 1]

**Problem 3.** Let \( R \) be an integral domain.
- (10 points) Define when an element \( x \in R \) is irreducible and when it is prime. Prove or give a counter-example: irreducible \( \Rightarrow \) prime, prime \( \Rightarrow \) irreducible.
- (10 points) Define when \( R \) is Euclidean and show that if \( R \) is Euclidean then it is a principal ideal domain.
- (10 points) Show that if \( R \) is a principal ideal domain and \( x \in R \) then \( x \) is prime \( \iff \) \( x \) is irreducible.

**Problem 4.** Let \( p \) be a prime number and consider a polynomial \( f_p(x) = x^4 + p^2 \in \mathbb{Q}[x] \)
- (10 points) Find the splitting field \( E \) of \( f_p \).
- (10 points) Find the Galois group of \( E \) over \( \mathbb{Q} \).
- (10 points) Is \( f_p \) irreducible in \( \mathbb{Q}[x] \)?

[The answers might depend on \( p \)]