1. Let $A$ be a commutative ring with 1 and let $M$ be an $A$-module.
   a. (5 points) Define what it means to say that $M$ is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.
   b. (10 points) When $A = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$ explain whether or not $M$ is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.
   c. (10 points) When $A = \mathbb{Z}$ and $M = \mathbb{Q}$ explain whether or not $M$ is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.

2. Let $A$ be a commutative ring with 1 and let $M$ be an $A$-module.
   a. (5 points) Define what it means for the module $M$ to be a Noetherian module, an Artinian module, a finitely generated module.
   b. (10 points) Let $M$ be a Noetherian $A$-module and $\alpha : M \rightarrow M$ a module homomorphism. Prove that if $u$ is surjective then $u$ is an isomorphism.
   c. (10 points) Prove that if $M$ and $N$ are finitely generated $A$ modules then $M \otimes_A N$ is again a finitely generated $A$-module.

3. Let $G$ be a finite abelian group.
   a. (15 points) Show that the rational group algebra $\mathbb{Q}[G]$ is a product of fields.
   b. (10 points) In the cases $G = \mathbb{Z}/2\mathbb{Z}$ and $G = \mathbb{Z}/3\mathbb{Z}$ identify the fields that occur in $\mathbb{Q}[G]$.

4. Let $\alpha : V \rightarrow V$ be a linear transformation of a finite vector space over a field $F$.
   a. (10 points) Prove that $\alpha$ is diagonalizable (there exists a basis of $V$ such that the associated matrix of $\alpha$ is a diagonal matrix) if and only if the minimal polynomial of $\alpha$ splits into linear factors (in $F$) with no repeated roots.
   b. (15 points) Prove that if $\alpha$ is diagonalizable then $V$ is a semi-simple $F[T]$ module (where the action is induced via $\alpha$ in the customary manner, given $f \in F[T]$ and $v \in V$, $f \cdot v = f(\alpha)(v)$).