1. (10 points) Let $Y$ be the connected sum of two Klein bottles. That is, take two Klein bottles $K_0$ and $K_1$, remove a disk from each and then connect the two punctured Klein bottles with a cylinder $S^1 \times [0,1]$ along the boundaries of the removed disks via homeomorphisms. Compute $\pi_1(Y)$ and $H_4(Y)$.

2. (10 points) Show that a space $X$ is simply-connected if and only if all (continuous) maps $S^1 \to X$ are homotopic.

3. (10 points) Let $n \geq 1$ and let $x_0, x_1, x_2$ be three distinct points in $\mathbb{R}^n$. Let $X = \mathbb{R}^n \setminus \{x_0, x_1, x_2\}$. Compute $\pi_1(X)$ and $H_4(X)$ (your answer will depend upon $n$).

4. (20 points, 5 each) Let $Y$ be a space and $G$ a group acting on $Y$ such that the group homomorphism $G \to \text{Homeo}(Y, Y)$ is injective. Assume furthermore that the action of $G$ satisfies the property:

   for each $y \in Y$ there is an open neighborhood $U$ such that 
   
   \[ g_1(U) \cap g_2(U) \neq \emptyset \Rightarrow g_1 = g_2 \text{ for all } g_1, g_2 \in G. \]

   We let $Y/G$ be the quotient space of $Y$ obtained by identifying orbits ($y \sim y' \iff gy = y'$ for some $g \in G$). We let $p : Y \to Y/G$ be the quotient map.

   a. Prove that $p$ is a covering map.

   b. Prove that $Y$ is Hausdorff if and only if $Y/G$ is Hausdorff.

   c. Prove that if $Y$ is path connected than $Y/G$ is path connected. Give an example showing that the converse is not true (i.e. where $Y/G$ is path connected but $Y$ is not path connected).

   d. Prove that if $Y$ is simply-connected than $Y/G$ is semilocally simply-connected.