Problem 1 Let $R$ be a commutative ring with 1 and $M$ an $R$-module. Recall that $M$ is defined to be $R$-flat if for every ideal $I \subseteq R$ the sequence

$$0 \to I \otimes_R M \to R \otimes_R M$$

is left exact, i.e., the map $I \otimes_R M \to R \otimes_R M$ is injective.

a. (7 points) Prove that $\mathbb{Q}$, the field of rational numbers, is $\mathbb{Z}$-flat, where $\mathbb{Z}$ is the ring of integers.
b. (6 points) Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$ module.
c. (7 points) Is the quotient $\mathbb{Q}/\mathbb{Z}$ $\mathbb{Z}$-flat? Justify your answer.

Problem 2 Let $R$ be a commutative ring with 1.

a. (7 points) Define what an injective $\mathbb{Z}$-module is, and show that both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.
b. (7 points) For any $R$-module $M$ define the dual $M^*$ to be $M^* = \text{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$. The dual of a module is itself an $R$-module. Prove that for any module $M$ the map

$$\phi_M : M \to (M^*)^*,$$

defined by $\phi_M(x)(f) = f(x)$, for $x \in M, f \in M^*$, is injective.
c. (6 points) If $P$ is a projective module its dual $P^*$ is injective. Use this to prove that any $R$-module can be embedded into an injective $R$-module.

Problem 3

a. (8 points) Prove that an Artinian integral domain is a field.
b. (4 points) Show that any prime ideal in an Artinian ring is maximal.
c. (8 points) Let $R = \frac{\mathbb{Q}[x,y,z]}{(x^2,y^2,z^2)}$. Let $M$ be any finitely generated $R$-module. Prove that $M$ has finite length over $R$.

Problem 4 For parts a., b. $k'$ is a field extension of a field $k$. 
a. (7 points) Let $A$ be a $n \times n$ matrix over $k$. Prove that the invariants of $A$ are the same as over $k'$.

b. (5 points) Let now $A, B$ be two $n \times n$ matrices over $k$ and assume there is an invertible matrix $C'$ over the extension $k'$ such that $B = C'A(C')^{-1}$. Show that there is an invertible matrix $C$ over $k$ such that $B = CAC^{-1}$. (Hint: you can use part a.)

c. (8 points) Let $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. Find the minimal polynomial of $A$ over $\mathbb{Q}$. Show that $A$ is similar to $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

**Problem 5** Let $R$ be a commutative ring and $M$ an $R$-module.

a. (4 points) Define what it means to say that $M$ is a cyclic module, and what it means to say that $M$ is simple (=irreducible).

b. (4 points) Show that all simple modules are cyclic.

c. (5 points) It is not true that all cyclic modules are simple. Give an example of this situation.

d. (7 points) Let $C_n$ be the cyclic group of order $n > 0$ and let $R = \mathbb{C}[C_n]$ be the complex group algebra of $C_n$. Find all cyclic $R$-modules.