Comprehensive Exam in Topology
University of Illinois, January 2009

1. (25 points) Let $H_*$ denote singular homology with integer coefficients. Let $X$ be a space, and let

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_k \subseteq \cdots X$$

be a sequence of open subsets of $X$ such that $\bigcup_{k=1}^{\infty} U_k = X$. Let $z \in H_n X$. Using only the definition of singular homology, show that there is a $k$ such that

$$z \in \text{Image}(H_n U_k \to H_n X).$$

2. (25 points)
Show that a (not necessarily connected) space $X$ is simply connected if and only if every pair of continuous maps $f_0, f_1 : S^1 \to X$ are homotopic to each other.

3. (25 points)
Let $S$ be a regular hexagon in the plane with vertices $P_1, \ldots, P_6$ (listed counter-clockwise). Let $X$ be the closed subset of the plane enclosed by $S$. Let $Y$ be the quotient space of $X$ obtained by making the identifications:

- Identify the edge $P_1 P_2$ with the edge $P_2 P_3$.
- Identify the edge $P_4 P_3$ with the edge $P_1 P_6$.

(Note that by "identify edge $AB$ with $CD$", we mean that each point along $AB$ is identified with the corresponding point along $CD$, so that in particular, $A \sim C$ and $B \sim D$ in the quotient.)

(a) Describe $\pi_1 Y$ in terms of generators and relations.
(b) Describe $H_1 Y$ in terms of generators and relations.

4. (25 points) Let $X = S^1 \times S^1$, and let $A = \{(a, b) \in S^1 \times S^1 \mid a = b\}$. Compute the relative homology groups $H_*(X, A)$. 