Comprehensive Exam: Differentiable Manifolds
August 2011

Problem 1 (15 points)
Show that the subset
\[ \{(x, y) \in \mathbb{R}^2 \mid x^3 + xy + y^3 = 1 \} \]
is a submanifold of \( \mathbb{R}^2 \).

Problem 2 (35 points)
(a) Let \( x, y, z \) be the standard coordinate functions on \( \mathbb{R}^3 \). Consider the vector fields
\[
X = x \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + 2 \frac{\partial}{\partial z},
\]
\[
Y = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}
\]
and the 2-form
\[
\omega = (z - y)dx \wedge dy + (x + y + z)dy \wedge dz
\]
on \( \mathbb{R}^3 \). Compute the following quantities.
(i) The time-\( t \) flow, \( \phi_t \), of the vector field \( X \).
(ii) The push forward map \( (\phi_1)_* : T_{(x,y,z)} \mathbb{R}^3 \to T_{\phi_1(x,y,z)} \mathbb{R}^3 \).
(iii) The Lie bracket \( [X, Y] \).
(iv) The exterior derivative \( d\omega \).
(b) (i) For a smooth vector field \( Y \) and a smooth \( k \)-form \( \omega \) on a smooth manifold \( M \) define the Lie derivative \( \mathcal{L}_Y \omega \)
(ii) For \( Y \) and \( \omega \) as in part (a), compute \( \mathcal{L}_Y \omega \).

Problem 3 (30 points)
(i) Let \( W \) be a compact oriented manifold of dimension \( k + 1 \) with nonempty boundary \( \partial W = M \). Let \( F : M \to N \) be a smooth map and \( \omega \) a smooth \( k \)-form on \( N \) such that \( d\omega = 0 \). Prove that if \( F \) can be extended to a smooth map \( \tilde{F} : W \to N \), then
\[
\int_M F^* \omega = 0.
\]
(ii) For $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ consider the smooth map $F : S^1 \to S^1$ defined by

$$F(x, y) = (-x, -y).$$

Prove that $F$ cannot be extended to a smooth map $\widetilde{F} : \mathbb{B}^2 \to S^1$ where $\mathbb{B}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

**Hint.** Consider part (i) and the one-form $\omega$ on $S^1$ defined as the restriction of

$$\left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$