Comprehensive exam in Topology (525)
August, 2011.

1. (25 points) Let $X$ be the space obtained from an annulus $\{ p \in \mathbb{R}^2 \mid 1 \leq |p| \leq 2 \}$ by identifying each point $(x, y)$ on the inner circle of radius 1 with the point $(-2x, -2y)$ on the outer circle of radius 2.

(a) Describe a CW-structure for $X$.
(b) Compute $\pi_1(X)$.
(c) Compute $H_*(X)$.

2. (25 points) Let $X = S^1 \times S^1$ (a torus). Classify all the 3-sheeted covering spaces over $X$, up to equivalence of covering spaces; give justification. (For the purposes of this question, a covering space of $X$ need not be a connected space.)

3. (25 points) Let $X$ be a path connected and locally path connected space $X$, and let $Y = S^1 \times \cdots \times S^1$, a product of $n$ copies of the circle, with $n \geq 1$.

Show that if $\pi_1(X, x_0)$ is finite, then every map $f : X \to Y$ is null-homotopic.

4. (25 points) Let $(X, A)$ be a pair, $A \neq \emptyset$, and write $j : A \to X$ for the inclusion map. Let $Y = X \cup_A CA$ the space obtained as a quotient of $X \amalg CA$ by identifying $i(a)$ with $j(a)$ for each $a \in A$.

(a) Show that $X$ is a retract of $Y$ if and only if $j$ is homotopic to a constant map.
(b) Show that $H_*(X, A) \approx H_*(Y, CA)$, where the map is induced by the inclusion $X \subset Y$.
(c) Show that $H_*(Y, CA) \approx \tilde{H}_*(Y)$, and thus $H_*(X, A) \approx \tilde{H}_*(Y)$. (Here, $\tilde{H}$ denotes reduced homology.)