Comprehensive Exam in Algebra (500)
August, 2012.

Each question is worth 25 points.

1. (a) Let \( H, K \) be subgroups of a group \( G \). Show that if \( H \trianglelefteq G \), then \( HK \) is also a subgroup of \( G \).

   (b) Given an example to show that \( HK \) can fail to be a subgroup if neither \( H \) or \( K \) are normal.

   (c) Prove that if \( H \trianglelefteq G \) of prime index \( p \), then for any subgroup \( K \leq G \) either (i) \( K \leq H \) or (ii) \( G = HK \) and \( |K : K \cap H| = p \).

2. (a) Let \( P \) be a \( p \)-Sylow subgroup of a finite group \( G \). Show that if \( N \trianglelefteq G \) is a normal subgroup of \( G \), then \( P \cap N \) is a \( p \)-Sylow subgroup of \( N \).

   (b) Give an example to show that (a) can fail if \( N \) is not normal.

   (c) Show that every group of order 460 = 4 · 5 · 23 is solvable.

3. Let \( R \) be an integral domain.

   (a) Given an element \( x \in R \) define what it means for \( x \) to be irreducible, and what it means for \( x \) to be prime. By proof or counterexample, determine whether irreducible implies prime, and whether prime implies irreducible.

   (b) Show that if \( R \) is a PID then \( x \in R \) is prime if and only if it is irreducible.

   (c) Let \( A \) denote the ring \( \mathbb{Z}(\sqrt{-5}) = \{ a + b\sqrt{-5} \mid a, b \in \mathbb{Z} \} \). Prove that \( A \) is not a principal ideal domain.

4. Let \( E = \mathbb{Q}(a) \) where \( a = \sqrt{1 + \sqrt{2}} \).

   (a) Find the irreducible polynomial of \( a \).

   (b) What is \( (E : \mathbb{Q}) \).

   (c) Identify the Galois group of \( E/\mathbb{Q} \).

   (d) How many subfields of \( E \) are there?