Comprehensive Exam
Fall 2012

Do your best. We are as much interested in your ability to think and reason as the correct answers.

1. [40 points] Let \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \) be a filtration of sub-sigma-algebras of \( \mathcal{F} \). Recall that for any stopping time \( \tau \),

\[
\mathcal{F}_\tau \overset{\text{def}}{=} \{ A \in \mathcal{F} : A \cap \{ \tau \leq n \} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N} \}.
\]

(a) [20 points] Suppose that \( \tau_1 \) and \( \tau_2 \) are two stopping times such that \( \tau_1 \leq \tau_2 \). Show that \( \mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2} \).

(b) [20 points] Show that \( \tau \) is \( \mathcal{F}_\tau \) measurable.

2. [50 points] Let’s construct the Prohorov metric on the collection \( \mathcal{P}(\mathbb{R}) \) of Borel probability measures on \( \mathbb{R} \). Let \( \mathcal{C} \) be the collection of closed subsets of \( \mathbb{R} \), and for any subset \( A \) of \( \mathbb{R} \) and any \( \varepsilon > 0 \), define

\[
A^\varepsilon \overset{\text{def}}{=} \{ x \in \mathbb{R} : \text{dist}(x, A) < \varepsilon \},
\]

where \( \text{dist}(x, A) \overset{\text{def}}{=} \inf_{y \in A} |x - y| \) for all \( x \in \mathbb{R} \). For any \( \mu \) and \( \nu \) in \( \mathcal{P}(\mathbb{R}) \), define

\[
\rho(\mu, \nu) \overset{\text{def}}{=} \inf \{ \varepsilon > 0 : \mu(F) \leq \nu(F^\varepsilon) + \varepsilon \text{ for all } F \in \mathcal{C} \}.
\]

Fix two points \( x \) and \( y \) in \( \mathbb{R} \).

(a) [25 points] Directly show that \( \rho(\delta_x, \delta_y) \leq |x - y| \).

(b) [25 points] Directly show that \( |x - y| \leq \rho(\delta_x, \delta_y) \).