LOGIC COMPREHENSIVE EXAM, AUGUST 2012

Each problem is worth 20 points for a total of 100 points.

Throughout, $L$ is a (first-order) language, and for a set $\Sigma$ of $L$-sentences, $\text{Mod}(\Sigma)$ is the class of models of $\Sigma$.

Problem 1. Let $\Sigma_1, \Sigma_2$ be sets of $L$-sentences such that no $L$-structure is a model of both $\Sigma_1$ and $\Sigma_2$. Show that there is an $L$-sentence $\sigma$ such that $\text{Mod}(\Sigma_1) \subseteq \text{Mod}(\sigma)$ and $\text{Mod}(\Sigma_2) \subseteq \text{Mod}(\neg \sigma)$.

Problem 2. Suppose $\sigma$ and $\tau$ are $L$-sentences and no non-logical symbol occurs in both $\sigma$ and $\tau$. Suppose also that every model of $\sigma$ is infinite and every model of $\neg \tau$ is infinite. Finally, suppose that $\sigma \rightarrow \tau$ is true in all $L$-structures. Prove that either $\neg \sigma$ is true in all $L$-structures or $\tau$ is true in all $L$-structures.

Problem 3. Suppose that $L$ has just a unary relation symbol $P$ and a binary relation symbol $\prec$. Let $T$ be the theory whose models are the structures $\mathcal{A} = (A, P^A, \prec^A)$, where $(A, \prec^A)$ is a dense linear order without endpoints and $P^A$ is a non-empty subset of $A$ such that whenever $b \in P^A$ and $a \prec^A b$, then $a \in P^A$. Find all complete $L$-theories extending $T$ by indicating for each such complete extension $T'$ a sentence $\sigma'$ such that $T \cup \{\sigma'\}$ axiomatizes $T'$.

Problem 4. Suppose the only non-logical symbol of $L$ is a binary predicate $R$. Consider the $L$-structure

$$\mathcal{A} := (\mathbb{Z}, \{(a, b) \in \mathbb{Z}^2 : a^2 = b^2\}).$$

(a) For which $k \in \mathbb{Z}$ is the singleton $(k)$ 0-definable in $\mathcal{A}$?

(b) For which infinite cardinals $\kappa$ is $\text{Th}(\mathcal{A})$ $\kappa$-categorical? (To be done without using results in model theory beyond Math 570.)
Problem 5. Let $L$ contain (at least) the constant symbol 0 and the unary function symbol $S$. Let $\Sigma$ be a set of $L$-sentences.

(a) Define what it means for a function $f: \mathbb{N} \to \mathbb{N}$ to be representable (as a function) in $\Sigma$.

(b) Suppose that $f, g: \mathbb{N} \to \mathbb{N}$ are functions that are representable in $\Sigma$ and $h = g \circ f$ is their composition. Show that $h$ is representable in $\Sigma$. 