U. OF ILLINOIS COMBINATORICS COMPREHENSIVE
EXAM – FALL 2012

Submit exactly THREE problems from each Part. Passing requires
good performance on each Part. Give full explanations, including
CLEAR STATEMENTS of any theorems that you use. Formulas with-
out explanations do not receive much credit.

No external assistance permitted.

PART I

1. Use known identities to simplify and evaluate

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{n + 1 - k} \]

2. A standard Young tableau of shape \((n, n)\) is a bijective filling of the
boxes of a Ferrers diagram of for the partition \((n, n)\) by the numbers
1, 2, \ldots, 2n such that the labels increase along rows and columns. For
example, if \(n = 3\), the five tableaux are:

\[
\begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\quad \begin{array}{cccc}
1 & 3 & 4 \\
2 & 5 & 6 \\
\end{array}
\quad \begin{array}{cccc}
1 & 2 & 5 \\
3 & 4 & 6 \\
\end{array}
\quad \begin{array}{cccc}
1 & 3 & 5 \\
2 & 4 & 6 \\
\end{array}
\quad \begin{array}{cccc}
1 & 2 & 4 \\
3 & 5 & 6 \\
\end{array}
\]

Prove that if \(SYT(n, n)\) is the set of all such tableaux then

\[ \#SYT(n, n) = C_n := \frac{1}{n + 1} \binom{2n}{n} \]

3. By a combinatorial construction with Ferrers diagrams, prove the
following identity of generating series:

\[ \prod_{k \geq 1} \left( 1 + zx^k \right) = 1 + \sum_{t \geq 1} \frac{x^t x^{(t+1)/2}}{\prod_{k=1}^{t} (1 - x^k)} \]

4. Let the set \(A_n\) be the set of permutations on \(\{1, 2, \ldots, n\}\) such that
in one line notation, \(i + 1\) does not appear, for \(1 \leq i \leq n - 1\). Let \(B_n\)
be the set of permutations \(\pi\) on \(\{1, 2, \ldots, n\}\) such that \(\pi(i) \neq i + 1\) for
\(1 \leq i \leq n - 1\). Prove that \(\#A_n = \#B_n\).

(For example, if \(n = 3\), then \(A_n = \{132, 213, 321\}\) whereas \(B_n = \{123, 321, 312\}\).)

PART II

5. Prove that a graph \(G = (V, E)\) with at least 6 vertices is 3-connected
if and only if for arbitrary disjoint subsets \(A, B\) of \(V\) with \(|A|, |B| \geq 3\),
there exist three fully disjoint paths from \(A\) to \(B\).
6. Prove König-Egervary Theorem on vertex covers using Hall’s Theorem.

7. Prove that each 3-regular graph with at most two cut edges has a perfect matching.

8. Present an example of a 4-connected graph that is not 2-linked. What about 5-connected graphs?

**PART III**

9. Prove that every set of \( n \) integers has a subset summing to a multiple of \( n \).

10. Prove or disprove:
    (a) Every graph \( G \) with chromatic number \( k \) has a proper \( k \)-coloring in which some color class has \( \alpha(G) \) vertices, where \( \alpha(G) \) denotes the independence number of \( G \).
    (b) If \( G \) is a connected graph, then the chromatic number of \( G \) is at most \( 1 + a(G) \), where \( a(G) \) is the average of the vertex degrees of \( G \). In other words, \( a(G) = 2|E(G)|/|V(G)| \).

11. Present an example of a tree and an ordering of its vertices such that the greedy coloring of this tree according to this ordering needs 5 colors.

12. Suppose \( H \) is a hypergraph with \( m \) edges in which every edge has size at least \( k \) and intersects fewer than \( 2^{k-1}/m \) other edges. Prove that \( H \) is 2-colorable.