MATH 500 — AUGUST 2015

Four problems, 25 points each. Maximum 100 points.

1. Let $G$ be a group of order $5 \cdot 13 \cdot 43 \cdot 73$. Determine the number of elements of order five.

2. Let $G$ be a finite group and $N$ a normal subgroup of $G$. Prove or disprove:
   (a) $G$ is nilpotent if and only if both $N$ and $G/N$ are nilpotent.
   (b) $G$ is solvable if and only if both $N$ and $G/N$ are solvable.

3. Let $R$ be an integral domain. Let $f \in R[x]$ be a nonzero polynomial such that there exist a nonzero $d \in R$ and polynomials $g, h \in R[x]$ of degree less than $f$ such that $df = gh$.
   (a) Show that if $R$ is a unique factorization domain then $f$ is the product of two polynomials in $R[x]$ of degree less than $f$.
   (b) Use part (a) with $f = x^2 - 5$ to show that $\mathbb{Z}[\sqrt{20}]$ is not a unique factorization domain.

4. Let $G$ be a finite group. Show that there exist fields $L$ and $K$ such that $L$ is an extension of $K$ with Galois group $G$. 