Solve five of the following six problems. Each problem is worth 20 points. Calculators, books and notes are not allowed. Good Luck!

**Notation:** \( m \) is the Lebesgue measure on \( \mathbb{R} \).

1. Let \( (a_j)_{j=1}^{\infty} \) be a sequence of complex numbers. Prove that

\[
\lim_{q \to \infty} \left( \sum_{j=1}^{n} |a_j|^q \right)^{1/q} = \sup_{j \in \{1, \ldots, n\}} |a_j|
\]

provided that \( n \in \mathbb{N} \) is independent of \( q \).

2. Show that

\[
\lim_{l \to \infty} \int_{0}^{b} \frac{\sin x}{x} \, dx = \frac{\pi}{2},
\]

by integrating \( e^{-xy} \sin x \) with respect to \( x \) and \( y \). **Justify your answer.**

3. Let \( f \) be a measurable function on a measure space \( (X, \mathcal{A}, \mu) \). The *decreasing rearrangement* of \( f \) is the function \( f^*: (0, \infty) \to [0, \infty] \) defined by

\[
f^*(t) = \inf\{\alpha : \mu(\{x : |f(x)| > \alpha\}) \leq t\},
\]

where \( \inf \emptyset \) is defined to be \( \infty \). Prove that

(a) \( f^* \) is decreasing;
(b) if \( f^*(t) < \infty \), then \( \mu(\{x : |f(x)| > f^*(t)\}) \leq t \).

4. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function and \( g : [0, 1] \to [0, 1] \) be a Borel function. Given \( n \in \mathbb{N} \), explain why the function \( f / |g(x)|^n \) is Lebesgue integrable over the closed interval \([0, 1] \). Compute the limit

\[
\lim_{n \to \infty} \int_{0}^{1} f / |g(x)|^n \, dx.
\]

**Justify your answer.**

5. If \( E \subseteq [0, 1] \), then \( \chi_E : [0, 1] \to \mathbb{R} \) denotes the characteristic function of \( E \). Find necessary and sufficient condition for the set \( E \subseteq [0, 1] \) to be Riemann integrable over the closed interval \([0, 1] \). **Justify your answer.**

6. Let \( E \subseteq \mathbb{R} \) be a non-empty Lebesgue measurable set and \( (f_n : E \to \mathbb{R})_{n=1}^{\infty} \) be a sequence of Lebesgue measurable functions. We say that the sequence \( (f_n)_{n=1}^{\infty} \) converges almost uniformly to a measurable function \( f : E \to \mathbb{R} \) on the set \( E \) as \( n \to \infty \) if for each \( \varepsilon > 0 \), there is a non-empty Lebesgue measurable set \( E_\varepsilon \subseteq E \) such that \( m(E_\varepsilon) < \varepsilon \) and the sequence \( (f_n)_{n=1}^{\infty} \) converges to \( f \) uniformly on the set \( E'_\varepsilon = E \setminus E_\varepsilon \) as \( n \to \infty \). Suppose that \( (f_n)_{n=1}^{\infty} \) is a Cauchy sequence in Lebesgue's measure \( m \), that is, for every \( \varepsilon > 0 \),

\[
\lim_{n,j \to \infty} m \{ x \in E : |f_n(x) - f_j(x)| \geq \varepsilon \} = 0.
\]

Prove that there is a Lebesgue measurable function \( f : E \to \mathbb{R} \) and that the sequence \( (f_n)_{n=1}^{\infty} \) has a subsequence \( (f_{n_k})_{k=1}^{\infty} \) such that \( (f_{n_k})_{k=1}^{\infty} \) converges almost uniformly to \( f \) on the set \( E \) as \( k \to \infty \).