Problem 1. (25 points) Suppose $A$ is a commutative ring with 1. Suppose $T$ is a subset of $A$ that satisfies the properties: i) $0 \not\in T$, ii) $1 \in T$, and iii) If $s, t \in T$, then their product $st \in T$. Prove that an ideal $M$ of $A$ maximal with respect to the property that $M \cap T = \emptyset$ is a prime ideal.

Problem 2. (25 points) Let $k$ be a field. Suppose $G, H$ are groups and let $k[G], k[H]$ be the group algebras over $k$. Prove that there is an isomorphism

$$k[G \times H] \cong k[G] \otimes_k k[H].$$

Problem 3. Let $A$ be a finite Abelian group.

a. (10 points) Show that the rational group algebra $\mathbb{Q}[A]$ is a product of fields.

b. (15 points) In case $A = \mathbb{Z}/2\mathbb{Z}$ and $A = \mathbb{Z}/3\mathbb{Z}$ identify the fields that occur in $\mathbb{Q}[A]$.

Problem 4. Let $A$ now be a commutative ring with 1 and $M$ an $A$-module.

a. (5 points) Define what it means to say that $M$ is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.

b. (10 points) In case $A = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z}$ explain whether or not $M$ is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.

c. (10 points) In case $A = \mathbb{Z}$ and $M = \mathbb{Q}$ explain whether or not $M$ is a cyclic module, a simple module, a semisimple module, a projective module, an injective module.