COMPREHENSIVE EXAM IN TOPOLOGY

JANUARY 2007

On this exam, $D^n$ is the standard $n$-disk

$$D^n = \{ x \in \mathbb{R}^n \| x \| \leq 1 \},$$

and $S^{n-1} = \{ x \in \mathbb{R}^n \| x \| = 1 \}$ is its boundary. For example, $S^0$ is the discrete space $\{ \pm 1 \}$.

1. Recall that $H_0 X \cong \mathbb{Z} \pi_0 X$. That is, the zeroth homology of $X$ is the free abelian group generated by the set of path components of $X$. So an element of $H_0 X$ can be written in the form

$$\sum a_i C_i,$$

where $a_i \in \mathbb{Z}$ and $C_i \in \pi_0 X$. Define the reduced homology $\hat{H}_0 X$ to be the kernel of the homomorphism

$$\epsilon : H_0 X \rightarrow \mathbb{Z}$$

given by

$$\epsilon(\sum a_i C_i) = \sum a_i.$$ 

Calculate $\hat{H}_0 S^0$, and calculate the effect on $\hat{H}_0 S^0$ of the map

$$r : S^0 \rightarrow S^0$$

given by the formula $r(x) = -x$.

2. In this problem $n$ is an integer greater than 1. Suppose that $(Y, \ast)$ is a pointed space, that

$$f : S^{n-1} \rightarrow Y$$

is a pointed map, and that $X$ is obtained from $Y$ by attaching an $n$-cell along $f$. That is, $X$ is the quotient

$$X = \frac{D^n \sqcup Y}{S^{n-1} \ni a \sim f(a) \in Y}.$$ 

Equivalently, $X$ is the pushout in the diagram

$$\begin{array}{ccc}
S^{n-1} & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
D^n & \longrightarrow & X.
\end{array}$$

For each $n \geq 2$, describe the relationship between $\pi_1(Y; \ast)$ and $\pi_1(X; \ast)$.

3. Let $p = (1, 0) \in S^1$. Let $T$ be the torus $S^1 \times S^1$, and let $W$ be the subspace $(S^1 \times \{ p \}) \cup (\{ p \} \times S^1) \subset T$. Show that $T/W \cong S^2$. Is the projection map

$$\pi : T \rightarrow S^2$$

homotopic to a constant map? Justify your answer.

4. In the following picture, is there a retraction $r : X \rightarrow A$? Is there a retraction $r : X \rightarrow B$? Justify your answers.

\[ X \] is a surface of genus 2.