Math 540 Comprehensive Examination
January 24, 2007

Solve all four problems in Part I and choose one problem in Part II. Indicate your choice. Credit will be given only for one problem in Part II. Each problem is worth 20 points.

\[ m \text{ denotes the Lebesgue measure on } \mathbb{R.} \]

Part I

1. Denote by \([x]\) the integer part of \(x\). Consider the function

\[ f : [0, 1] \to \mathbb{R}, \quad f(x) = \sum_{n=1}^{\infty} \frac{[nx]}{3^n}. \]

(a) Prove that \(f\) is Lebesgue integrable and compute its integral on \([0, 1]\), expressing the answer in the form of a rational number.

(b) Is \(f\) Riemann integrable on \([0, 1]\)? Justify your answer.

2. Decide whether each of the following three statements is true or false. Justify your answer (i.e. provide the reason and a short proof if true and a counterexample if false).

(a) Let \((f_n)\) be a sequence of measurable functions \(f_n : \mathbb{R} \to \mathbb{R}\) such that \(f_n \to f\) pointwise on \(\mathbb{R}\). Then there exists a subsequence \((f_{n_k})\) such that \(f_{n_k} \to f\) in measure.

(Recall that, by definition, \(f_n \to f\) in measure when \(m(\{|f_n - f| > \delta\}) \to 0\) for every \(\delta > 0\).)

(b) If \(f : [a, b] \to \mathbb{R}\) is continuous on \([a, b]\) and \(f'\) is bounded (a.e.) on \((a, b)\), then \(f\) must be absolutely continuous.

(c) Let \(1 \leq p < \infty\) and \(f \in L^p(\mathbb{R})\). Then

\[ \forall \varepsilon > 0, \exists N > 0 \text{ such that } m(\{|f| > N\}) < \varepsilon. \]

3. Let

\[ f(x) = \int_{1}^{\infty} \frac{e^{-xy}}{y^3} \, dy, \quad x > 0. \]

Show that \(f\) is differentiable on \((0, \infty)\) and find a formula for \(f'\).

4. Suppose that \(B\) is a measurable subset of \([0, 1]\). Define

\[ g(x) = m((\infty, x) \cap B). \]

Prove that \(g'\) exists almost everywhere and calculate \(\int_{[-1,2]} g' \, dm\) explaining your reasoning.
Part II

5. Let \( f : [0, 1] \to [0, \infty) \) in \( L^1 \) such that

\[
(*) \quad \int_E f \, dm \leq \sqrt{m(E)} \quad \text{for every } E \subseteq [0, 1] \text{ measurable.}
\]

(a) Prove that \( f \in L^p[0, 1] \) for all \( p \in [1, 2) \).

(b) Give an example of a function \( f \) which satisfies (*) but \( f \notin L^2[0, 1] \).

6. In a Hilbert space \( \mathcal{H} \) consider an orthonormal set \( (u_n)_{n=1}^{\infty} \).

(a) Prove Bessel's inequality

\[
(**) \quad \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2, \quad \forall x \in \mathcal{H}.
\]

(b) Use part (a) to show that if, in addition, \( \{u_n : n \geq 1\} = \{0\} \), then equality holds in (**).