Math 531 Comprehensive Exam
January 2008

Problem 1
(a) Find an asymptotic formula for the sum \( \sum_{p \leq x} \frac{\log^2 p}{p^{1/2}} \), where the sum is over primes \( p \).
(b) Let \([a_1, a_2, \ldots, a_n]\) be the least common multiple of \( a_1, a_2, \ldots, a_n \). Find the radius of convergence of the series
\[ \sum_{n=1}^{\infty} \frac{z^n}{[1, 2, \ldots, n]} \cdot \]

Problem 2
Suppose \( S(x) = \sum_{n \leq x} a_n \sim Lx \), where \( a_1, a_2, \ldots \) are real numbers and \( L \) is a nonzero real number. Consider the Dirichlet series
\[ F(s) = \sum_{n=1}^{\infty} a_n n^{-s} \cdot \]

(a) Show that \( F(s) \) has abscissa of convergence \( \sigma_c = 1 \).
(b) Find an asymptotic formula for \( \sum_{n \leq x} \frac{a_n}{n} \).

Problem 3
(a) Prove that \( \tau(n^2) \) is multiplicative, where \( \tau(m) \) is the number of positive divisors of \( m \).
(b) Prove that
\[ \sum_{n=1}^{\infty} \frac{\tau(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)} \quad (\text{Re } s > 1). \]

Hint: use the identity \( \sum_{m=0}^{\infty} (2m+1)x^m = \frac{1+x}{(1-x)^2} \) for \( |x| < 1 \).
(c) Assuming the Riemann Hypothesis, describe as accurately as possible the location of the poles of the right side of the equation in (b).
Problem 4
(a) Use the multiplicative properties of the Möbius function $\mu$ to show that
\[ \sum_{d^2 \mid n} \mu(d) = \begin{cases} 1 & \text{n is squarefree} \\ 0 & \text{otherwise} \end{cases}. \]

(b) If $n$ has prime factorization $n = p_1^{e_1} \cdots p_r^{e_r}$, let $\lambda(n) = (-1)^{e_1 + \cdots + e_r}$. Find an asymptotic formula for
\[ \sum_{n \leq x} \sum_{\substack{\lambda(n) \neq \mu(n) \leq x}} 1. \]

Problem 5
Let $\chi$ be a real, nonprincipal character modulo $k$.
(a) Show that there are infinitely many primes $q$ with $\chi(q) = 1$ and infinitely many primes $q$ with $\chi(q) = -1$.

(b) Determine whether or not the series $\sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}}$ converges or diverges.