Justify your answers. Good luck.

**Problem 1** [10 points]. Find the number of symmetries of a cube (i.e. the number of permutations of the 8 vertices which take edges to edges and faces to faces).

**Problem 2.** Let $H$ and $K$ be normal subgroups of a group $G$. Assume that $HK = G$ and $H \cap K = \{1\}$.

a [10 points]. Prove that $hk = kh$ for any $h \in H$ and $k \in K$.

b [10 points]. Prove that $G$ is isomorphic to $H \times K$.

**Problem 3** [10 points]. Let $P$ be a Sylow $p$-subgroup of a finite group $G$ and $N$ be a normal subgroup of $G$. Prove that $P \cap N$ is a Sylow $p$-subgroup of $N$.

*Hint: you might want to consider the subgroup $PN/N$ of $G/N$.*

**Problem 4.**

a [10 points]. Define an Euclidean domain and a PID. Prove that an Euclidean domain is a PID.

b [10 points]. Prove that $\mathbb{Q}[x]$ is an Euclidean domain and hence a PID.

c [10 points]. Find a generator of the ideal $\langle x^3 - 3x + 2, x^4 - 1, x^6 - 1 \rangle$ of $\mathbb{Q}[x]$.

**Problem 5.** Consider $F = \mathbb{Q}[\sqrt{5}]$ as an extension of $\mathbb{Q}$.

a [10 points]. Find the Galois group of $F$ over $\mathbb{Q}$.

b [10 points]. Find the normal closure $E$ of $F$.

c [10 points]. Find the Galois group of $E$ over $\mathbb{Q}$.