Math 531 Comprehensive Exam
January 2009

Problem 1
(i) State, without proof, an asymptotic formula for \( \psi(x) = \sum_{n \leq x} \Lambda(n) \).

(ii) State, without proof, an estimate for \( M(x) = \sum_{n \leq x} \mu(n) \) which is equivalent to the prime number theorem.

(iii) Find, with proof, an asymptotic formula for \( \sum_{p \leq x} \frac{\sqrt{\log p}}{p} \).

Problem 2
Let \( a > 0 \). Define an arithmetic function \( k_a \) by

\[
\zeta(s)\zeta(s-a) = \sum_{m=1}^{\infty} \frac{k_a(m)}{m^s} \quad (\text{Re } s > a + 1).
\]

Here \( \zeta \) is the Riemann zeta function.

(i) Give a combinatorial or number-theoretic interpretation for \( k_a(m) \).

(ii) Prove that \( k_a \) is a multiplicative function.

(iii) Let \( a > 1 \). Find an asymptotic formula for

\[
\sum_{m \leq x} k_a(m)
\]

with relative error \( O(1/x) \) (that is, show that the sum is \( f(x)(1 + O(1/x)) \) for some “simple” function \( f \)).

Problem 3
(i) Give an infinite product expansion for the Gamma function \( \Gamma(z) \).

(ii) State the location of all zeros and poles of \( \Gamma(z) \).

(iii) State the functional equation relating \( \zeta(s) \) to \( \zeta(1-s) \).

(iv) Let \( \text{Re } s > 1 \). Express

\[
I(s) = s \int_0^\infty e^{-st} \psi(e^t) dt
\]

in terms of \( \zeta(s) \).

(iv) State, as precisely as possible, the location of the poles of the meromorphic continuation of \( I(s) \) to the complex plane. Be sure to say whether \( I(s) \) has finitely many or infinitely many poles.
Problem 4

(i) Show that there are infinitely many primes whose last four digits are 2009.

(ii) Let \( q > 1 \). Determine the limit

\[
\lim_{s \to 1^+} \sum_{n \equiv 1 \pmod{q}} \frac{\mu(n)}{n^s}.
\]

Be sure to justify why the limit exists.