COMPREHENSIVE EXAM, MATH 500, JANUARY 20, 2010.

1. (a) Let $P$ be a Sylow $p$-subgroup of a finite group $G$ and let $N_G(P) \leq H \leq G$ where $N_G(P)$ denotes the normalizer. Prove that $N_G(H) = H$.
(b) Show that there are no simple groups of order 616.

2. Let $G$ be a group with a composition series of finite length, $l(G)$, and let $N \triangleleft G$.
   (a) Show that $N$ has a composition series.
   (b) Show that $G/N$ has a composition series.
   (c) Prove that $l(G) = l(N) + l(G/N)$.

3. Let $E$ be the field extension $\mathbb{Q}(\sqrt{3} - \sqrt{2})$, where $\mathbb{Q}$ is the rational field.
   (a) Prove that $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
   (b) Find $(E : \mathbb{Q})$.
   (c) Find the irreducible polynomial of $\sqrt{3} - \sqrt{2}$.
   (d) If $K$ is a field such that $\mathbb{Q} \leq K \leq E$, prove that $K$ is normal over $\mathbb{Q}$.

4. (a) Prove that for any commutative ring $R$ and a proper ideal $I$ in $R$, there exists a maximal ideal in $R$ containing $I$.
   (b) Prove that if $J$ is a maximal ideal in a commutative ring $R$, then the quotient $R/J$ is a field.