Solve any four of the following five problems. Only four problems will be graded. Clearly indicate which problem is not to be graded. Each problem is worth 25 points. Justify all your answers. Good Luck!

\[ D = \{ z \in \mathbb{C} \mid |z| < 1 \} \]
\[ \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \} \]

**Notation:** \( H(G) \) denotes the set of all analytic functions on an open set \( G \subseteq \mathbb{C} \)

1. Let \( \Omega \) be an open set in \( \mathbb{C} \) containing the closed unit disk \( \overline{D} \) and \( f \in H(\Omega) \). Let \( \Gamma \) be a closed curve in \( \mathbb{C} \) defined by \( w(t) = f(e^{it}), 0 \leq t \leq 2\pi \). Prove that \( \ell(\Gamma) \geq 2\pi |f'(0)| \), where \( \ell(\Gamma) \) is the length of \( \Gamma \).
   **Hint:** first prove that \( \ell(\Gamma) = \int_0^{2\pi} |f'(e^{it})| \, dt \).

2. Let \( f \) be a meromorphic function in a neighborhood of \( \overline{D} \) with zeros \( z_1, z_2, \ldots, z_n \in D \) and poles \( p_1, p_2, \ldots, p_m \in D \) (counted according to multiplicity) and analytic on the boundary of \( D \). Suppose that \( \text{Im}(f(z)) \neq 0 \) for every \( z \) on the boundary of \( D \). Prove that \( m = n \).

3. Let \( P_n \) be a polynomial of degree \( n \). For each \( r > 0 \), set
   \[ M(r) = \max_{|z|=r} |P_n(z)|. \]
   (a) Prove that
   \[ \frac{M(r_1)}{r_1^n} \geq \frac{M(r_2)}{r_2^n} \]
   whenever \( 0 < r_1 < r_2 \).
   (b) Suppose that there are positive \( r_1 \) and \( r_2 \), \( r_1 \neq r_2 \), such that \( \frac{M(r_1)}{r_1^n} = \frac{M(r_2)}{r_2^n} \). Prove that there is \( a \in \mathbb{C} \) such that \( P_n(z) = az^n \).

4. Let \( \Omega \subseteq \mathbb{C} \) be a connected open set and let \( (f_n)_{n=1}^{\infty} \) be a sequence of 1-1 analytic functions in \( \Omega \) converging uniformly on each compact subset of \( \Omega \) to a function \( f \). Prove that \( f \) is either constant or 1-1 function in \( H(\Omega) \).

5. Find the constants \( c_n, n = 0, 1, 2, \ldots \), such that
   \[ \frac{\pi}{\sin(\pi z)} = \frac{c_0}{z} + \sum_{n=1}^{\infty} \frac{c_n z}{z^2 - n^2}, z \in \mathbb{C} \setminus \mathbb{Z} \]
   and prove that the series \( \frac{c_0}{z} + \sum_{n=1}^{\infty} \frac{c_n z}{z^2 - n^2} \) converges uniformly to \( \frac{\pi}{\sin(\pi z)} \) on each compact subset of \( \mathbb{C} \) after removing finitely many terms.

   **Hint:** Use Liouville's theorem to prove the equality; you can use without proof that the right-hand side of the above equality is periodic with period \( p = 2 \) and also use without proof the following inequality \( \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \leq \frac{\pi}{2a}, a > 0 \).